Soc Choice Welfare (1995) 12: 267–276

Social Choice and Welfare

A derivation of the money rawlsian solution*

Enriqueta Aragones**

MEDS-KGSM, Northwestern University, Evanston, Il 60208, USA
(Fax: 708 467 12 20)

Received: 23 December 1992/Accepted: 25 November 1994

Abstract. We study the set of envy-free allocations for economies with indivisible objects and quasi-linear utility functions. We characterize the minimal amount of money necessary for its nonemptiness when negative distributions of money are not allowed. We also find that, when this is precisely the available amount of money, there is a unique way to combine objects and money such that these bundles may form an envy-free allocation. Based on this property, we describe a solution to the envy-free selection problem following a pseudo-egalitarian criterion. This solution coincides with the "Money Rawlsian Solution" proposed by Alkan et al. (1991).

1. Introduction

Envy-free allocations, as definied by Foley (1967) are allocations for which every agent prefers his own bundle to those assigned to other agents. It is well known that they do not always exist when there are indivisible goods to be allocated among the agents. In order to guarantee the existence of envy-free allocations for these economies, there has to exist an infinitely divisible good (that we may think of as money) to compensate agents when the distribution of the indivisible ones generates envy among them.

The availability of the infinitely divisible good (money) guarantees the existence of envy-free allocations if we do not have any restriction on the distribution of money (Alkan et al. 1991). Sometimes, we may want the money allocations to be

* I am indebted to I. Gilboa for his valuable suggestions and his patience during the elaboration of the final version. I also wish to thank S. Barbera for his guidance in an earlier version and M. Boldrin, H. Moulin, Z. Neeman, W. Thomson and the referees for their comments. Financial support from FPU-MEC (Spain) is gratefully acknowledged.

** Northwestern University.

nonnegative, since the allocation of negative quantities of money requires additional assumptions on the original wealth of the agents. In this case, envy-free allocations only exist when the quantity of money available in the economy is large enough (Alkan et al. 1991; Maskin 1987).

The economies we deal with are defined by a set of objects, a quantity of money and a set of agents with preferences defined on objects and money. Each agent receives one and only one object and an amount of money in addition to the wealth they may already have. A typical example is the allocation of jobs and money compensations, where money comes from taxes paid by the agents. In this case we may have special interest in solutions that choose allocations using the minimal possible amount of money. We address here the problem of deciding in which way the goods and the money should be allocated among the agents if all agents are supposed to have the same property rights over the objects and the money (see Moulin 1993). First we are interested in allocations that do not generate any envy, but for these economies the set of envy-free allocations may be quite large. Thus we are faced with a selection problem. Since a solution to this problem that chooses many different allocations is not very useful, the goal of this paper is to find a solution that gives a precise recommendation of how to allocate objects (indivisible goods) in a way that is envy-free.

In this paper, we analyze the structure of the set of envy-free allocations. The existence of envy-free allocations with indivisible goods, quasi-linear preferences and no restrictions on the sign of the distribution of money can be proved from linear programming duality or by a direct combinatorial argument (see Alkan et al. 1991). Here, we present a constructive proof that reveals the special structure of the set of envy-free allocations and will also suggest a natural solution to the selection problem. We characterize the minimal amount of money that guarantees the existence of envy-free allocations with nonnegative distribution of money. We use this result to define a solution to the envy-free selection problem.

This problem has been addressed by Alkan et al. (1991), and Tadenuma and Thomson (1991a, b) who proposed solutions based on intuitive considerations of fairness. We approach the selection problem by applying a pseudo-egalitarian criterion and the resulting solution coincides with the money Rawlsian solution proposed by Alkan et al. (1991), and is based on a different principle (the maximin principle). First, we construct an envy-free allocation that allocates nonnegative quantities of money to the agents and uses the minimal amount of money. We find that this allocation is "almost unique" in two ways: first, there is only one way of combining objects and money to have envy-freeness; second, all agents are indifferent among all envy-free allocations (Tadenuma and Thomson (1993) refer to this property as "single-valuedness up to indifferent permutations"). When there is more than one envy-free allocation, the same bundles are given to different agents. This implies that when we have just enough money to guarantee the existence of envy-free allocations the envy-free solution gives as precise a recommendation to the allocation problem as one could hope for (obviously, ties can always be broken arbitrarily or randomly). When money is given exogenously and exceeds the amount needed to solve the envy-free problem, we allocate it equally and we retain the egalitarian and envy-free properties as well as the uniqueness of the utility profile. We describe a procedure to calculate the set of allocations selected by our solution. The computation of these allocations can be done by an algorithm of a polynomial time complexity. The technique used in the construction of the envy-free allocation and in the characterization of the minimal amount of money is similar to the one used in Quinzii (1984) to find the prices of a competitive equilibrium for an exchange economy.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 studies the set of envy-free allocations with and without restrictions on the distribution of money. Finally, Section 4 describes how to construct the solution to the selection problem.

2. The model

An economy is represented by an ordered pair $e = (F, M)$, where M is a real number representing the amount of an infinitely divisible good, which we call money. F describes the fundamentals of economy e and is given by an ordered triple $F = (Q, A, u_Q)$, where $Q = \{1, 2, \dots, n\}$ is a finite set of agents, $A = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a finite set of objects. Each agent $i \in Q$ is endowed with a preference relation defined on the product of A and the real line, which is assumed to admit a numerical representation by a quasi-linear utility function:

$$U_i(\alpha_j, x) = u_i(\alpha_j) + x.$$

This function is interpreted as the utility that agent $i \in Q$ derives when he receives an object $\alpha_j \in A$ and an amount of money $x \in \Re$. The symbol u_Q, in the economy description, stands for a list of n nonnegative vectors, one for each agent $i \in Q$.

$$u_Q = \{[u_i(\alpha_1), u_i(\alpha_2), \dots, u_i(\alpha_n)]\}\ i \in Q.$$

We will denote by ε the class of such economies. Given an economy $e = (F, M) \in \varepsilon$, an *allocation* is a pair $z = (\sigma, m)$, where σ is a bijection, $\sigma: Q \to A$, assigning to each agent $i \in Q$ an element $\alpha_i \in A$, and where $m = \{m_1, m_2, \dots, m_n\}$ is a vector in R^n and m_i is to be thought of as the amount of money agent i receives. We say that an allocation $z = (\sigma, m)$ is *feasible* when the total amount of money is distributed among the agents, i.e.:

$$\sum_{i=1}^{n} m_i = M.$$

$Z(e)$ will denote the set of feasible allocations for the economy $e \in \varepsilon$.

In some cases it may be of little interest to consider allocations in which agents receive negative quantities of money, unless we make additional assumptions (e.g., the agents hold positive quantities of money). Therefore, we also study the case in which agents can only recieve nonnegative quantities of money. Obviously, only economies with a nonnegative total quantity of money M are of interest in this case. The sub-class of ε containing these economies will be called ε_+, i.e., $\varepsilon_+ = \{e = (F, M) \in \varepsilon \mid M \geq 0\}$.

Correspondingly, we define a modified concept of feasibility: an allocation $z = (\sigma, m) \in Z(e)$ where $e = (F, M) \in \varepsilon_+$, is called *feasible with nonnegative transfers* if it is feasible and $m \geq 0$. $Z_+(e)$ will denote the set of feasible allocations with nonnegative transfers for the economy $e \in \varepsilon_+$. Naturally, for each economy $e \in \varepsilon_+$, $Z_+(e) \subseteq Z(e)$.

3. Envy-free allocations

In order to analyze the set of envy-free allocations for our economies we give a characterization for Pareto efficient allocations and we use the already known results of existence and efficiency of envy-free allocations for our economies. The proof of existence of envy-free allocations with nonnegative distributions of money for our economies reveals the special structure of the set $E(e)$, and will also suggest a natural solution to the selection problem. See Svensson (1983) and Alkan et al. (1991) for general proofs.

Definition (Foley 1967). An allocation $z = (\sigma, m) \in Z(e)$, where $e = (F, M) \in \varepsilon$, is called *envy-free* if

$$u_i(\sigma(i)) + m_i \geq u_i(\sigma(j)) + m_j \quad \text{for all } i, j \in Q.$$

$E(e)$ will denote the set of envy-free allocations for the economy $e \in \varepsilon$.

Given $e \in \varepsilon$, define an *optimal assignment of objects* for e as a bijection $\sigma: Q \to A$ such that:

$$\sum_{i=1}^{n} u_i(\sigma(i)) \geq \sum_{i=1}^{n} u_i(\sigma'(i)) \quad \text{for every bijection } \sigma': Q \to A.$$

Define $P^*(e) = \{z = (\sigma, m) \in Z(e) \mid \sigma \text{ is an optimal assignment of objects for } e\}$.

Since for each economy $e = (F, M) \in \varepsilon$, there is a finite number ($n!$) of assignments of objects among agents, there will always exist at least one satisfying the condition above. Therefore, the set $P^*(e)$ is never empty. And this property characterizes the set of Pareto efficient allocations for the class of economies we are interested in. $P(e)$ will denote the set of Pareto efficient allocations for $e \in \varepsilon$. The proofs of the following results are not included but they can be easily checked.

Proposition 1. *For all $e \in \varepsilon$, $P^*(e) = P(e)$ and $E(e) \subseteq P(e) = P^*(e)$.*

The proof of this result for a general class of preferences can be found in Svensson (1983). When we restrict our attention to the set of feasible allocations with nonnegative transfers, we still have existence for allocations with optimal assignments of objects but this property no longer characterizes the Pareto efficient allocations. The sets of Pareto optimal and Envy-free allocations, and allocations with optimal assignments of objects with respect to $Z_+(e)$ are denoted respectively by $P_+(e)$, $E_+(e)$, and $P^*_+(e)$. In this case we have the following result.

Proposition 2. *For all $e \in \varepsilon_+$, $P^*_+(e) \subseteq P_+(e)$ and $E_+(e) \subseteq P^*_+(e)$.*

Hence, only optimal assignments of objects can generate envy-free allocations. To show the existence of envy-free allocations we need some additional notation: Let k^{σ}_{ij} denote the extent to which agent i envies agent j given the distribution of objects among agents σ, i.e.,

$$k^{\sigma}_{ij} = u_i(\sigma(j)) - u_i(\sigma(i)).$$

(Note that this expression may, of course, be negative.)

Given $F = (Q, A, u_Q)$ and a distribution of objects $\sigma: Q \to A$, we construct a weighted graph $G_\sigma = (Q, Q^2)$, where every agent is represented by a node and every two agents are connected by an arc. We define the weight of arc (i, j) to be k^{σ}_{ij}.

The total weight of a directed path $(r, T) = [r(1), \ldots, r(T)]$, with $r(t) \in Q$ for each $t = 1, \ldots, T$, is given by

$$w(r, T) = \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

A cycle is a directed path (r, T) with $r(1) = r(T)$. A path with $T = 1$ is called a loop.

Lemma 1. *An assignment of objects σ is optimal if and only if every cycle in G_σ has a nonpositive total weight.*

Proof. We know that σ is an optimal assignment of objects if and only if:

$$\sum_{i=1}^{n} k^{\sigma}_{i\sigma^{-1}(\sigma'(i))} \leq 0 \quad \text{for every bijection } \sigma': Q \to A.$$

Given σ', we can define a set of orbits of $\sigma^{-1}(\sigma')$ in the graph G_σ. Hence, σ is an optimal assignment of objects if and only if the sum of the total weights of these cycles is nonpositive. We want to prove that this is true if and only if the weight of every cycle in the graph is nonpositive. However, the "if" is immediate and the "only if" is simple: Suppose one of this cycles has positive total weight, i.e., for some $i \in Q$

$$\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} > 0, \quad \text{where } r(1) = r(T) = i.$$

Supplementing r by $(n - T)$ trivial cycles $\{(i, i)\}_{i \notin \{r(j)\}_{j=1}^{T}}$ yields a contradiction. Hence σ is an optimal assignment of objects if and only if all cycles are nonpositive. □

Lemma 2. *If σ is an optimal assignment of objects, then for each $i \in Q$*

$$\max\left\{\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \quad s.t.\ r(1) = i \text{ and } (r, T) \text{ is any path in } G_\sigma\right\}$$

has a finite and nonnegative solution.

Proof. First, we show that for every path (r, T) there is another path (r', T') with $T' \leq n$ such that $w(r', T') \geq w(r, T)$. If $T > n$ then (r, T) must contain a nontrivial cycle, because the graph has only n nodes. For each cycle there exist t^* and t^{**} such that $t^* < t^{**} \leq T$ and $r(t^*) = r(t^{**})$. Therefore, the total weight of (r, T) can be decomposed as follows:

$$w(r, T) = \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} = \sum_{t=1}^{t^*-1} k^{\sigma}_{(t)r(t+1)} + \sum_{t=t^*}^{t^{**}-1} k^{\sigma}_{r(t)r(t+1)} + \sum_{t=t^{**}}^{T-1} k^{\sigma}_{r(t)r(t+1)}.$$

Since $[r(t^*), r(t^* + 1), \ldots, r^*(t^{**})]$ is a cycle, optimality implies that its total weight must be non positive. Then

$$w(r, T) = \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \leq \sum_{t=1}^{t^*-1} k^{\sigma}_{r(t)r(t+1)} + \sum_{t=t^*}^{T-1} k^{\sigma}_{r(t)r(t+1)}.$$

Following this reasoning for all cycles, we shall find a path (r', T') with $T' \leq n$ contained in the original path (r, T). Therefore for each $i \in Q$ there is a path (r_i, T_i) with $T_i \leq n$, which satisfies:

$$w(r_i, T_i) = \max\left\{\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i \text{ and } (r, T) \text{ is any path in } G_\sigma\right\}$$

$$= \max\left\{\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i \text{ and } (r, T) \text{ is any path in } G_\sigma \text{ with } T \leq n\right\}.$$

Since the solution for the right hand side problem exists and is finite, we also have a finite solution to our problem. This solution must be nonnegative, since loops (r, T), with $w(r, T) = 0$, are also in the feasible set. □

To prove the existence of envy-free allocations we start by showing that, given the fundamentals of an economy, for every optimal assignment of objects we can find a nonnegative distribution of money such that they form an envy-free allocation for the economy defined by these fundamentals and the amount of money given by the total amount distributed.

Theorem 1. *Let $F = (Q, A, u_Q)$, and $\sigma: Q \to A$ be an optimal assignment of objects for F. Define:*

$$m_i^* \equiv \max\left\{\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i \text{ and } (r, T) \text{ is any path in } G_\sigma\right\},$$

$$m^*(\sigma) \equiv (m_i^*(\sigma))_{i \in Q},$$

$$M^*(\sigma) \equiv \sum_{i=1}^{n} m_i^*(\sigma).$$

Then, $(\sigma, m^*(\sigma)) \in E_+(F, M^*(\sigma))$.

Proof. Given $F = (Q, A, u_Q)$, fix an optimal assignment of objects $\sigma: Q \to A$ and consider an allocation $z = (\sigma, m^*(\sigma))$, where for each $i \in Q$

$$m_i^*(\sigma) = w(r_i, T_i) = \max\left\{\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i \text{ and } (r, T) \text{ is any path in } G_\sigma\right\}.$$

By Lemma 2 we know that this maximization problem has a finite and nonnegative solution. The argument below shows that $z = (\sigma, m^*(\sigma))$ is an envy-free allocation for the economy $e = (F, M^*(\sigma))$. By Lemma 2 we know that $m_i^*(\sigma) \geq 0$ for all $i \in Q$. Suppose we had

$$m_i^*(\sigma) - m_j^*(\sigma) = \sum_{t=1}^{T_i - 1} k^{\sigma}_{r_i(t)r_i(t+1)} - \sum_{t=1}^{T_j - 1} k^{\sigma}_{r_j(t)r_j(t+1)} < k^{\sigma}_{ij}.$$

This inequality implies that there is a path $[i, r_j(1), \ldots, r(T_j)]$ starting at i with a higher total weight than (r_i, T_i), which is a contradiction. This yields:

$$m_i^*(\sigma) - m_j^*(\sigma) = \sum_{t=1}^{T_i - 1} k^{\sigma}_{r_i(t)r_i(t+1)} - \sum_{t=1}^{T_j - 1} k^{\sigma}_{r_j(t)r_j(t+1)} \geq k^{\sigma}_{ij} \quad \text{for all } i.$$

Whence $z = (\sigma, m^*(\sigma)) \in E(F, M^*(\sigma))$. □

The existence of envy-free allocations with nonnegative transfers can only be guaranteed when the total quantity of money in the economy is large enough. Maskin (1981) and Alkan (1991) give sufficient conditions on the total quantity for a more general class of economies. For our particular model we have a necessary and sufficient condition on the total quantity of money for envy-free allocations with nonnegative transfers to exist. This result is stated in Theorem 2 and it follows directly from the next two lemmata.

Lemma 3. *Let* $F = (Q, A, u_Q)$, and $\sigma: Q \to A$ *be an optimal assignment of objects for* F. *Let* $m = (m_1, \dots, m_n) \in \Re^n$. *If* $(\sigma, m) \in E(F, \sum_{i=1}^n m_i)$ *and for some* $i \in Q$, $m_i < m_i^*(\sigma)$, *then for some* $j \in Q$, $m_j < 0$.

Proof. Suppose $(\sigma, m) \in E(F, \sum_{i=1}^n m_i)$ and $m_i < m_i^*(\sigma)$ for some $i \in Q$. Since (σ, m) is envy-free, $m_h - m_j \geq k_{hj}^{\sigma}$ must hold for all $h, j \in Q$. This implies:

$$\begin{aligned} m_i < m_i^* = \sum_{t=1}^{T_i - 1} k^{\sigma}_{r_i(t) r_i(t+1)} &\leq m_{r_i(1)} - m_{r_i(2)} + m_{r_i(2)} - m_{r_i(3)} \\ &+ \cdots + m_{r_i(T_i - 1)} - m_{r_i(T_i)} \\ &= m_{r_i(1)} - m_{r_i(T_i)} = m_i - m_{r_i(T_i)} \end{aligned}$$

which, in turn, implies that $m_{r_i(T_i)} < 0$. □

Lemma 3 shows that given an optimal assignment of objects the amount of money found by the construction of the allocation in Theorem 1 is minimal for the existence of envy-free allocations with nonnegative distribution of money, and that given the same optimal assignment of objects and this amount of money the envy-free allocation is unique.

Corollary. *Let* $F = (Q, A, u_Q)$, *and* $\sigma: Q \to A$ *be an optimal assignment of objects for* F.

(i) *If* $M < M^*(\sigma)$, *then there is no* $(\sigma, m) \in Z_+(F, M)$ *such that* (σ, m) *is envy-free.*
(ii) *There is no* $m \in \Re^n$ *such that* $m \neq m^*(\sigma)$ *and* $(\sigma, m) \in E_+(F, M^*(\sigma))$.

Lemma 4 shows that when there are several optimal assignments of objects, we can use the same distribution of money to construct the envy-free allocations with nonnegative transfers.

Lemma 4. *Let* $F = (Q, A, u_Q)$, *and* $\sigma: Q \to A$ *be two optimal assignments of objects for* F. *Define* $m' \in \Re^n$ *as* $m_i' \equiv m^*(\sigma)_{\sigma^{-1}(\sigma'(i))}$ *for all* $i \in Q$. *Then,* $(\sigma', m') \in E_+(F, M^*(\sigma))$.

Proof. Suppose (σ', m') is not envy-free. Then, there eixst $i, h \in Q$ such that

$$u_i(\sigma'(i)) + m_i' < u_i(\sigma'(h)) + m_h'.$$

Denoting $j = \sigma^{-1}(\sigma'(h))$ and by definition of z'

$$u_i(\sigma'(i)) + m_i' = u_i(\sigma'(i)) + m_{\sigma^{-1}(\sigma'(i))} < u_i(\sigma(j)) + m_j = u_i(\sigma'(h)) + m_h'.$$

Rearranging terms and using the definition of k_{ij}^{σ} we have

$$\begin{aligned} m_{\sigma^{-1}(\sigma'(i))} - m_j < u_i(\sigma(j)) - u_i(\sigma'(i)) &= u_i(\sigma(j)) - u_i(\sigma(i)) \\ + u_i(\sigma(i)) - u_i(\sigma'(i)) &= k_{ij}^{\sigma} - k_{i\sigma^{-1}(\sigma'(i))}^{\sigma}. \end{aligned}$$

Because both σ and σ' are optimal assignments $\sum_{h=1}^{n} k^{\sigma}_{h\sigma^{-1}(\sigma'(h))} = 0$, and we can write

$$k^{\sigma}_{ij} - k^{\sigma}_{i\sigma^{-1}(\sigma'(i))} = k^{\sigma}_{ij} + \sum_{h=1;h\neq i} k^{\sigma}_{h\sigma^{-1}(\sigma'(h))}.$$

Since $z = (\sigma, m)$ is envy-free we have

$$\begin{aligned} k^{\sigma}_{ij} + \sum_{h=1;h\neq i}^{n} k^{\sigma}_{h\sigma^{-1}(\sigma'(h))} &\leq k^{\sigma}_{ij} + \sum_{h=1;h\neq i}^{n} [m_h - m_{\sigma^{-1}(\sigma'(h))}] \\ &= k^{\sigma}_{ij} + M^*(\sigma) - m_i - M^*(\sigma) + m_{\sigma^{-1}(\sigma'(i))} \\ &= k^{\sigma}_{ij} - m_i + m_{\sigma^{-1}(\sigma'(i))}. \end{aligned}$$

We have $m_{\sigma^{-1}(\sigma'(i))} - m_j < k^{\sigma}_{ij} - m_i + m_{\sigma^{-1}(\sigma'(i))}$ which implies that $m_i - m_j < k^{\sigma}_{ij}$ and contradicts the fact that (σ, m) is envy-free. Therefore, we have that $(\sigma', m') \in E_+(F, M^*(\sigma))$ □

From Lemma 4 we obtain the minimal amount of money for the existence of envy-free allocations with nonnegative distributions of money and we can show that when this is the amount of money available all agents are indifferent among all envy-free allocations.

Corollary. *Let $F = (Q, A, u_Q)$, and $\sigma: Q \to A$ and $\sigma': Q \to A$ be two optimal assignments of objects for F.*

(i) $M^*(\sigma) = M^*(\sigma')$.
(ii) *For all* $\alpha \in A$, $m^*(\sigma')_{\sigma'^{-1}(\alpha)} = m^*(\sigma)_{\sigma^{-1}(\alpha)}$.
(iii) *For all* $i \in Q$, $u_i(\sigma(i)) + m^*(\sigma)_i = u_i(\sigma'(i)) + m^*(\sigma')_i$.

Proof. To prove part (i) we already have that $M^*(\sigma) \geq M^*(\sigma')$. Following the same reasoning for σ' we have $M^*(\sigma) \leq M^*(\sigma')$, hence $M^*(\sigma) = M^*(\sigma')$. Part (ii) follows from part (i). If (σ, m) is an envy-free allocation which uses the smallest amount of money, for each $i \in Q$ we must have that $u_i(\sigma(i)) + m_i = \max_{j\in Q}\{u_i(\sigma(j)) + m_j\}$. By the same reason $u_i(\sigma'(i)) + m_i = \max_{j\in Q}\{u_i(\sigma'(j)) + m_{\sigma^{-1}(\sigma'(j))}\}$, and by construction of (σ', m') these quantities are the same. This proves part (iii). □

Notice that if the money vector were indexed on the objects, instead of on the individuals, there would exist a unique envy-free allocation with nonnegative distribution of money when the amount of money available is minimal. Now we can state the main result.

Given $F = (Q, A, u_Q)$, define $M^*_F \equiv M^*(\sigma)$ for all optimal assignments σ for F.

Theorem 2. *For all $F = (Q, A, u_Q)$, $E_+(F, M) \neq \emptyset$ if and only if $M \geq M^*_F$.*

Proof. The "if" part follows directly from Theorem 1. The "only if" part follows from Corollary of Lemma 3 once we known that $M^*(\sigma) = M^*_F$ from Lemma 4. □

4. A solution to the envy-free selection problem

Now we want to define a solution to the envy-free selection problem based on the structure of the set of envy-free allocations studied in the last section.

Definition. A solution is defined as a correspondence Φ which assigns to each economy $e \in \varepsilon$ a non-empty subset $\Phi(e)$ of $Z(e)$.

We would like a solution to give precise recommendations regarding the distribution of objects and money among the agents. We have seen that in general there are a large number of envy-free allocations. We propose a solution which may be applied to all economies in our class, and will always choose a unique allocation. This solution is defined as follows:

$$\Phi^*(F, M) = \{(\sigma, m) \in E(F, M): m_i = m_i^* + (M - M_F^*)/n$$
$$\text{and } (\sigma, m^*) \in E(F, M_F^*)\}.$$

It allocates the money as follows: given any optimal assignment of objects each agent will receive the unique amount of money indicated by the envy-free allocation for (F, M_F^*) plus an equal share of the amount of money left over, i.e. it allocates the minimal amount of money to generate an envy-free allocation and when there is no more envy applies only the egalitarian criterion to allocate the rest of the money, and the final allocations are still envy-free.

Using Theorems 1 and 2 it is easy to check that this solution chooses at least one allocation for each economy and is single valued up to permutations which leave all agents indifferent. It can be shown that this solution coincides with "the money Rawlsian solution" proposed by Alkan et al. (1991) based on the maximin principle. The next proposition provides an algorithm to compute an allocation selected by this solution and shows that it is of polynomial time complexity.

Proposition 3. *There exists an algorithm, whose time complexity is polynomial, that computes an element of Φ^* and the utility profile corresponding to Φ^* for any given economy. (With rational data.)*

Proof. Given $e = (F, M)$ consider the following procedure to construct an element of the set $\Phi^*(F, M)$.

First step. Find the optimal assignments of objects. Given $F = (Q, A, u_Q)$, let $G_F = (Q \cup A, Q \times A)$ be a directed bipartite graph, where every agent and every object are represented by a node ($Q \cup A$ is the set of nodes) and each agent is connected to each object by an arc ($Q \times A = \{(i, \alpha_j): i \in Q \text{ and } \alpha_j \in A\}$ is the set of arcs). We define the weight of an arc (i, α_j) to be $u_i(\alpha_j)$. Then the problem of finding an optimal assignment of objects can be thought of as a weighted matching problem:

"Given an arc-weighted bipartite graph, find a matching for which the sum of the arcs is maximum."

This can be done in $O(n^3)$ steps. (See, for instance, Lawler (1976, p. 201–207.)

Second step. Find M^*. Following the proof of Theorem 1 $M^* = \sum_{i=1}^n m_i^*$, therefore we have to find, for each $i \in Q$:

$$m_i^* = \max\left\{\sum_{t=1}^{T-1} k^\sigma_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i \text{ and } (r, T) \text{ is any path in } G_\sigma\right\}$$

for any optimal assignment of objects σ. This problem can be written as $(n-1)$ different "shortest paths problems":

$$m_i^* = \max_{j \in Q}\left\{-\min \sum_{t=1}^{T-1} -k^\sigma_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i, (r, T) \in G_\sigma \text{ and } r(T) = j\right\}.$$

By lemma 1 we know that if σ is an optimal assignment of objects the graph G_σ has no cycles with positive weight. Given this condition a solution can be found in $O(n^3)$ operations. (See Lawler 1976, pp 82–89.) Once we have all the shortest paths between all pairs of nodes, we have to select, for each $i \in Q$, the shortest path from i, which requires $O(n^2)$ steps.

Third step. Compute an allocation in $\Phi^*(F, M)$. Given σ, compute $m_i^\sigma = m_i^* + (M - M^*)/n$ to obtain $z \equiv (\sigma, m_i^\sigma) \in \Phi^*(F, M)$. This can obviously be done in linear time.

Finally note that the time complexity of the algorithm is $O(n^3)$: step 1 requires $O(n^3)$ operations, step 2 requires $O(n^3)$ as well, while step 3 is of linear time complexity. □

References

Alkan A, Demange G, Gale D (1991) Fair allocation of indivisible objects and criteria of justice. Econometrica 59: 1023–1039

Foley D (1967) Resource allocation and the public sector. Yale Economic Essays 7: 45–98

Lawler E (1976) Combinatorial optimization: networks and matroids (Ed. Holt) Rinehart and Winston, New York

Maskin E (1987) On the fair allocation of indivisible goods. Arrow and the Foundations of the Theory of Economic Policy (Ed. G. Feiwel) London Macmillan

Moulin H (1992) An application of the Shapley Value to fair division with money. Econometrica, 60: 1331–1349

Quinzii M (1984) Core and Competitive Equilibria with Indivisibilities. Int J Game Theory 13: 41–60

Svensson L (1983) Large indivisibles: an analysis with respect to price equilbirium and fairness. Econometrica 51: 939–954

Tadenuma K, Thomson W (1991a) Non-envy and consistency in economies with indivisible goods. Econometrica 59: 1755–1767

Tadenuma K, Thomson W (1991b) Solutions to the problem of fair allocation in economies with indivisible goods. University of Rochester mimeo.

Tadenuma K, Thomson W (1993) The fair allocation of an indivisible good when monetary compensations are possible. Math Soc Sci 25: 117–132