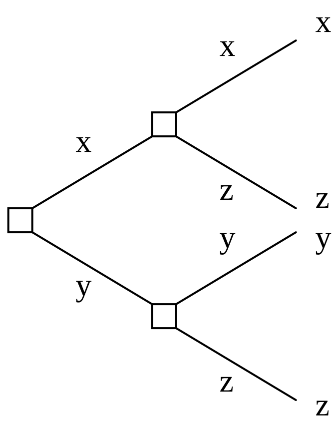
6. Strategic Voting and Agenda Control

A voter facing a choice between two alternatives has a dominant strategy: choose his most preferred alternative. A voter facing a choice among more than two alternatives has no dominant strategy: it's best strategy depends on the choice of the other voters.

Definitions:
A voter votes sincerely if he always votes for his most preferred alternative.
A voter votes strategically (sophisticated vote) if he always decides his vote optimally, taking into account its effects, and the other voters effects over the final outcome.

An agenda, is an ordered list of alternatives, for instance {x,y,z} that is interpreted as: first a vote is taken between x and y, and then a vote in takes between the winner and z. It may be represented with a tree.



Sincere Voting

Consider three alternatives: x, y, and z; and three voters: 1, 2, and 3 with preferences:

Voter 1	Voter 2	Voter 3
x	y	z
y	z	x
z	x	y

Consider the following agenda: first, voters have to decide between alternatives x and y, then voters have to decided between the winner and z.
If the three voters vote sincerely, we would have that:

- x wins against y
- z wins against x
- z is the winner

Now, consider a different agenda: first voters choose between y and z, then voters choose between the winner and x.

- y wins against z
- x wins against y
- x is the winner

Finally consider the following agenda: first voters choose between x and z, then voters choose between the winner and y.

- z wins against x
- y wins against z
- y is the winner

Now consider the following example with three alternatives and three voters with preferences:

Voter 1	Voter 2	Voter 3
x	y	z
y	z	y
z	x	x

Assume that the agenda is: first, x against y, and then the winner against z.
Next assume that the agenda is: first, x against z, and then the winner against y.
Last assume that the agenda is: first, z against y, and then the winner against x.
Notice that in the three cases, the outcome is alternative y. This is because y is a Condorcet winner.

McKelvey's Chaos Theorem (sincere voting):
If there is no Condorcet winner, then for any alternatives x and y there exists a finite agenda commencing at y and ending at x, $V = (y, z_1, \ldots, z_n, x)$ with z_1Py, z_iPz_{i-1} and xPz_n.

Where P refers to the result of majority voting.
Thus, from any alternative there is an agenda that would lead sincere voters to any other alternative.

McKelvey proved that for strong simple preference aggregation rules applied to multidimensional sets of alternatives, the typical situation is that either the core is nonempty or the top-cycle set includes all available alternatives.

Let a weak P-chain between x and y be a sequence $x_1, \ldots, x_k$ such that $x_1 = x$, $x_k = y$ and not $x_{j+1}Px_j$ $(x=x_1 \geq \ldots \geq x_k=y)$

TC(P) = {x: for all $y \in X$, there is a weak P-chain between x and y}

The Top Cycle is the set of alternatives that can reach any alternative in X via some weak T-chain.

Strategic Voting

Consider three alternatives: x, y, and z; and three voters: 1, 2, and 3 with preferences:

Voter 1	Voter 2	Voter 3
x	y	z
y	z	x
z	x	y

Consider the following agenda: first, voters have to decide between alternatives x and y, then voters have to decided between the winner and z.

If the three voters vote strategically :
Observe that if in the first round voter 1 chooses y instead of x, in the second round y would defeat z. In this case the winner would be y, and voter 1 is better off since he prefers y to z. Thus, voter 1 has a best reply that does not coincide with his sincere vote. When we consider all voters voting strategically the outcome of the game is given by its sub-game perfect equilibrium (SPE). And we can compute it by backward induction.

Observe that in the last round, sincere voting is a dominant strategy for all voters. Thus, we have that:

- z would win against x
- y would win against z

Thus, in the first round, voters will take into account the optimal results of the second round: they know that only z and y might end up winning.
When in the first round a voter is asked to choose between x and y, he will answer by choosing between z and y, and he will choose x if he prefers z to y; and he will choose y if he prefers y to z. Since there is a majority of voters that prefer y to z, in equilibrium y defeats x in the first round, and y defeats z in the second round. Thus, y is the outcome of the SPE. Observe that no voter has a profitable deviation in this case.

Now, consider a different agenda: first voters choose between y and z, then voters choose between the winner and x.
Since sincere voting is a dominant strategy in the second round we have that:

- x would win against y
- z would win against x

Thus, in the first round, voters will take into account the optimal results of the second round: they know that only z and x might end up winning.
When in the first round a voter is asked to choose between z and y, he will answer by choosing between z and x, and he will choose y if he prefers x to z; and he will choose z if he prefers z to x. Since there is a majority of voters that prefer z to x, in equilibrium z defeats y in the first round, and z defeats x in the second round. Thus, z is the outcome of the SPE. Observe that no voter has a profitable deviation in this case.

Finally consider the following agenda: first voters choose between x and z, then voters choose between the winner and y. Using a similar reasoning one can prove that the outcome of the SPE in this case is alternative x.

Therefore, we have seen that the order in which the alternatives appear in the agenda affects the final outcome. This is the reason that controlling agenda enables with power.

Observe that:
With the first agenda we obtain y, the most preferred alternative of voter 2.
With the second agenda we obtain z, the most preferred alternative of voter 1.
With the third agenda we obtain x, the most preferred alternative of voter 3.

If one of the voters can determine the agenda, he will choose the one that favors his most preferred alternative.

Now consider the following example with three alternatives and three voters with preferences:

Voter 1	Voter 2	Voter 3
x	y	z
y	z	y
z	x	x

Assume that the agenda is: first, x against y, and then the winner against z.
Next assume that the agenda is: first, x against z, and then the winner against y.
Last assume that the agenda is: first, z against y, and then the winner against x.
Notice that in the three cases, the SPE outcome is alternative y. This is because y is a Condorcet winner.

Shepsle and Weingast's Theorem (strategic voting):
If there is no Condorcet winner, there exists a finite agenda with y the first element and x the sophisticated agenda equilibrium, if and only if x is not covered by y.

Thus, from any alternative there is an agenda that would lead sophisticated voters to any other alternative not covered by the initial one.

Definitions:

The win set of x = the set of alternatives majority preferred to x:
$W(x) = \{y: yPx\}$

There is no Condorcet winner iff $W(x) \neq \varnothing$ for all x.

$W^{-1}(x) = \{y: xPy\}$
$W(x) \subset W(y)$ implies $W^{-1}(y) \subset W^{-1}(x)$

For all x and y, y covers x (yCx) iff

(i) $y \in W(x)$
(ii) $W(y) \subset W(x)$

Proof of Shepsle and Weingast's Theorem:

Sufficiency: Suppose that y does not cover x. Then either:

(1) $y \notin W(x)$ or

(2) (2) $y \in W(x)$ and $W(y) \not\subset W(x)$.

If $y \notin W(x)$ then agenda V = (y,x) yields x as the SPE outcome.
If $W(y) \not\subset W(x)$ and $y \in W(x)$ then there is a $z \in W(y)$ and $z \notin W(x)$, and agenda V = (y,x,z) yields x as the SPE outcome.
Since in the last round: z defeats y and x defeats z, in the first round voters choose between x and z, and x wins.

Necessity: Suppose that y covers x and there is an agenda V with x and y that yields x as the SPE outcome.
If x precedes y, since $y \in W(x)$, then x cannot be the SPE outcome.
If y precedes x, it can be shown by induction (see proof in the paper).

Uncovered set:
$UC(X) = \{x: yCx \text{ for no } y \in X\}$
$UC(y) = \{x: \neg yCx\}$
$UC(X) = \cap\, UC(y)$

Uncovered Set:
generalization of Condorcet Winner
if preferences are quasi concave then the UC set contains the equilibrium outcomes from:

two candidate competition in a large electorate
cooperative behavior in small committees
sophisticated behavior in a legislative environment

Pareto set:
$PO(X) = \{x: yP_ix \text{ for no } y \in X\}$
If preferences are Euclidean, then the Pareto set is the convex hull of the voters ideal points.

$UC(X) \subset PO(X)$
If preferences are Euclidean, then the uncovered set is a relatively small subset of the convex hull of ideal points.

Proof:
Suppose that $x \in UC(X)$ and $x \notin PO(X)$
Then there is y such that yP_ix for all i, thus $y \in W(x)$
Take any $z \in W(y)$, then zP_iy by a majority.
Thus, zP_ix by a majority: $z \in W(x)$
Therefore, $W(y) \subset W(x)$ and also $y \in W(x)$, which implies that yCx, therefore $x \notin UC(X)$.

Two dimensions characterize the set of agenda institutions:
1) sincere versus sophisticated voting agents.
2) centralized (closed) versus non-centralized (open to amendments) agenda.

If there is no Condorcet winner:

The set of equilibria given centralized agenda and sincere voting is X (McKelvey's theorem).
The set of equilibria given centralized agenda and sophisticated voting is UC(y) for any initial alternative y (Shepsle and Weingast's theorem).

Consider a completely decentralized agenda mechanism: any agent may add an alternative to the voting order.

The set of equilibria given this decentralized agenda and sincere voting is empty.
(Since $W(x) \neq \emptyset$ for all x, for any agenda with sincere outcome x, there is always a voter with incentive to introduce $y \in W(x)$ at the last round and produce y as the equilibrium outcome.
Thus, final outcomes in this case will depend on the chosen stopping rule, and the set of potential outcomes is X.)

The set of equilibria given this decentralized agenda and sophisticated voting is UC(X) (from Shepsle and Weingast's theorem).
Because if an alternative is not in the UC then it cannot be the outcome of a sophisticated agenda voting.

Banks Set:
The set of un-dominated sub-game perfect Nash equilibrium outcomes for committee voting over finite amendment agendas (or voting by successive elimination).
It defines bounds on the monopoly power of an agenda setter.

Amendment agendas and voting by successive elimination:
Given a finite number of alternatives, $x_1, \ldots, x_L$ the sophisticated outcome induced by ordering s is the last element, w_L, of the finite sequence described by:

$$w_1^s = x_1, and\ \ for\ all\ l > 1: w_l^s = \begin{cases} x_l\ \ if\ \ x_l P w_{l'}^s\ \ for\ all\ l' < l \\ \quad w_{l-1}^s\ otherwise \end{cases}$$

The Banks Set is the subset of alternatives which are sophisticated outcome for at least one ordering of X.

Let a P-chain between x and y be a sequence of alternatives $x_1, \ldots, x_k$ such that $x_1 = x$, $x_k = y$ and $x_j P x_{j+1}$ ($x=x_1 > \ldots > x_k = y$)
Let an x-chain of P be a P-chain where x beats all other alternative sin the chain according to P.

B(P) = {x: there is an x-chain of P such that for all y not in the chain there is a z in the chain such that nor yPx}

Thus, Banks showed that the outcomes found by varying the ordering (for a fixed tournament) of the amendment agenda when voting by successive elimination correspond the endpoints of chain, where the chains are such that any alternative not included in the chain in beaten by something in the chain. The intuition behind the characterization is that the alternatives in the chain are those that temporarily win at some stage in the voting, and the remaining alternatives are those who are eliminated at their stages.

In general:
If P is an asymmetric binary relation, then $CW \subseteq B \subseteq UC \subseteq TC$

References:
Banks, Jeffrey S. (1985) "Sophisticated Voting Outcomes and Agenda Control" *Social Choice and Welfare*, 1: 295-306.
Dutta, Bhaskar, Matthew O. Jackson and Michel Le Breton (2005) "The Banks Set and The Uncovered Set in Budget Allocation Problems" in *Social Choice and Strategic Decisions*, ed. David Austen-Smith and John Duggan, Berlin: Springer.
Ordeshook, Peter. 1986. "*Game Theory and Political Theory*" New York: Cambridge University Press.
Shepsle, Kenneth and Barry Weingast. 1984. "Uncovered sets and sophisticated voting outcomes with implications for agenda institutions" AJPS 28:49-74.