

Voters' private valuation of candidates' quality*

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Abstract

We propose a generalization of the model of electoral competition between two candidates who differ in non-policy characteristics that allows voters to have heterogeneous and private valuations of these characteristics. Unlike models in which voters' valuations of such non-policy characteristics are common and common knowledge and which *never admit pure strategy equilibria*, in our setup we show *existence of both converging and mildly diverging pure strategy equilibria*. Perhaps more importantly, we uncover a non-monotonic (U-shaped) relationship between the extent of heterogeneity of voters' preferences in the issue of candidates' non-policy characteristics and the maximum degree of equilibrium platform differentiation. Our analysis a) sheds new light on how a small (large) expected degree of heterogeneity in the voters' preferences on the candidates' non-policy characteristics may lead to a depoliticized (politicized) vote and b) demonstrates that as voters become more heterogeneous in this issue, existence of pure strategy equilibria becomes more likely.

Keywords: Downsian model; private information; advantaged candidate; platform differentiation.

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1 Introduction

In the existing models of electoral competition with a favored candidate or party it is assumed that all voters value one of the candidates' non-policy characteristics more than the opponent's in the same way and that this information is common knowledge by all actors of the model (see for example Ansolabehere and Snyder 2000, Groseclose 2001, Aragonès and Palfrey 2002, Ashworth and Bueno de Mesquita 2009 and Ansolabehere et al. 2012).¹ The common knowledge that all voters value the non-policy characteristics of a certain candidate more than the non-policy characteristics of the other one, makes candidates have very diverse incentives as far as electoral platform selection is concerned: the advantaged candidate always wants to *depoliticize* the electoral campaign - he wants to imitate the policy platform of the disadvantaged candidate so that only the non-policy characteristics will determine how voters will vote - while the disadvantaged candidate always aims at *politicize* the vote - he wants to offer a policy platform distinctly different from the one offered by the advantaged candidate and hence induce voters to vote also on the basis of which platform they like most. When candidates are Downsian (office motivated) these diverse dynamics lead to inexistence of pure strategy equilibria and, subsequently, of stable electoral competition outcomes. In the mixed strategy equilibria of these models the advantaged candidate proposes (in expected terms) more moderate policies than the disadvantaged one and elections are politicized - differentiation of policy platforms is sufficiently large and hence some voters vote for the disadvantaged candidate (Aragonès and Palfrey 2002, Hummel 2010 and Aragonès and Xefteris 2012).

In these Downsian models with an advantaged candidate, candidates are *uncertain* about the voters' preferences over policy and share common beliefs about them, but they are *certain* that *all* voters prefer the non-policy characteristics of a specific candidate. Two natural steps that would make these models more realistic would be a) to allow voters be heterogeneous, not only in policy terms, but also in how they evaluate the non-policy characteristics of candidates and b) to account for candidates having incomplete information in both dimensions - they should be uncertain about how each voter values candidates' non-policy characteristics as well. This is precisely the gap in the literature that this paper aims to fill.

We propose a generalization of the original model in which *different voters are allowed to have different valuations of candidates' non-policy characteristics*. In this set up a voter's valuation of the non-policy characteristics of the candidates is considered as the voter's *private information* and it is possibly different for each voter. Candidates share a common prior belief on how many voters prefer the non-policy

¹Further results on electoral competition between heterogeneous candidates (or parties) may be found in Stokes (1963), Erikson and Palfrey (2000), Dix and Santore (2002), Laussel and Le Breton (2002), Herrera et al. (2006), Schofield (2007), Degan (2007), Kartik and McAfee (2007), Carillo and Castanheira (2008), Meirowitz (2008), Zakharov (2009), Hummel (2010), Krasa and Polborn (2012), Xefteris (2012), Pastine and Pastine (2012) and Bernheim and Kartik (2014).

characteristics of one candidate over the other one. Formally, we represent the candidates' beliefs about their non-policy characteristic as a random variable with a Bernoulli distribution that is common for all voters and we assume that each voter's valuation is given by an independent random draw from it. That is, from the candidates' perspective, a voter prefers the non-policy characteristics of the first candidate with probability $\rho \in [\frac{1}{2}, 1]$. The assumption that this probability is always at least one-half is obviously without loss of generality and it allows us to address the first candidate by the name *advantaged candidate* simply because he is the one whose non-policy characteristics are expected to be valued more by most voters. In fact, in the model we propose the advantage has two dimensions. On the one hand, there is the magnitude of the difference between the qualities of the two candidates from a voter's point of view ($d > 0$) and on the other hand there is the probability with which the advantaged candidate enjoys the aforementioned advantage ($\rho > \frac{1}{2}$).

We find that in this model *pure strategy equilibria exist* for a wide range of parameter values - as long as voters are expected to be sufficiently heterogeneous ($\rho > \frac{1}{2}$ but close to $\frac{1}{2}$) in their preferences regarding candidates' non-policy characteristics. These pure strategy equilibria exhibit converging and mildly diverging pure strategies. Hence it is not the fact that voters have preferences about candidates' non-policy characteristics that rules out pure strategy equilibria but the assumption that preferences of the voters on this issue are common to all voters and they are common knowledge information. These pure strategy equilibria are such that elections are depoliticized: candidates offer sufficiently similar platforms and each voter votes for the candidate whose non-policy characteristics he values most. Within this range of parameter values, as the expected share of voters who find the first candidate better than the second increases (that is, as ρ increases), the maximum degree of equilibrium differentiation decreases. The most striking feature of this process is that this decrease in the maximum value of equilibrium differentiation occurs only because the set of equilibrium strategies of the first candidate (the advantaged one) shrinks around the centre of the policy space. That is, the set of equilibrium strategies of the second candidate (the disadvantaged one) remains invariant to changes in the expected size of the two groups of voters as long as the group which thinks he is better is sufficiently large (that is, as long as $1 - \rho$ is sufficiently large). When the expected sizes of the two groups become very asymmetric ($\rho > \frac{1}{2}$ and close to 1) pure strategy equilibria cease to exist and a unique mixed equilibrium exists such that the advantaged candidate locates in the centre of the policy space and the disadvantaged mixes between two policies which are equidistant from the centre of the policy space. This mixed equilibrium is such that elections are politicized: in expected terms some voters vote for the candidate whose non-policy characteristics they value less only because they like the policy he proposed much more than the one of their favored candidate. For these parameter values, as the expected share of voters who find the first candidate better

than the second increases (that is, as ρ increases), the maximum degree of equilibrium differentiation *increases*. Again, what is most striking is that this is only because the two policies that are part of the disadvantaged candidate's mixed strategy go farther away from the centre of the policy space while the equilibrium behavior of the advantaged candidate remains unaffected.

This shows that when a candidate starts to become advantaged, he most probably moves towards the centre while at the same time the set of equilibrium strategies of the disadvantaged candidate remains invariant. It is actually the move of the advantaged candidate towards the centre that eliminates incentives of the disadvantaged one to politicize the elections: the closer the advantaged candidate is to the centre the farther away from the centre the disadvantaged candidate would have to locate to politicize the elections. At some point though, when the expected share of voters who value the non-policy characteristics of the first candidate becomes much larger than the expected share of voters who think that the second candidate is better, the disadvantaged candidate is better off by politicizing the elections independently of where the advantaged candidate locates. From that point on the equilibrium is such that the advantaged candidate locates precisely at the centre and the disadvantaged one drifts slowly away as the share of voters who find his non-policy characteristics better decreases. In the limit, that is when $\rho \rightarrow 1$, we converge to the equilibrium of the complete information model (see, for example, Aragonès and Xefteris, 2012). These asynchronous effects of an increase in $\rho \in [\frac{1}{2}, 1]$ on equilibrium strategies - first only the advantaged candidate moves towards the centre and then only the disadvantaged one drifts away - are responsible for a non-monotonic relationship between the extent of heterogeneity of voters' preferences in the issue of candidates' non-policy characteristics and the maximum degree of equilibrium platform differentiation. An increasing asymmetry in how non-policy characteristics of candidates are viewed by the voters first decreases differentiation between candidates' platforms but after a critical point it pushes these platforms farther and farther away.

Since we have introduced two novelties (*heterogeneous* and *private* valuation of candidates non-policy characteristics) in this paper a natural question is the following: to which of these two novelties one should mostly attribute existence of pure strategy equilibria? Are both of them equally necessary for pure strategy equilibria to exist or, maybe, one of them could be enough? To provide an answer, one should notice first that when voters have *heterogeneous* valuations of candidates' non-policy characteristics which are *common knowledge* (that is, candidates know which one is a majority's favored candidate), it is impossible to have pure strategy equilibria: the majority's favored candidate can win with certainty if he offers the same policy as the other candidate and hence the other candidate has to mix in order to differentiate from the favored candidate and to maintain some positive probability of election. Then one should study the case in which valuations are *homogeneous* but *private information* (that is, all voters like

the same candidate but the candidates do not know who this candidate is). Existence or inexistence of pure strategy equilibria in this case is not obvious. For this reason we formally investigate this variation of the model and we find that indeed pure strategy equilibria exist even in this case. This shows that it is the assumption of private information, rather than heterogeneity of voters' preferences on the issue of candidates' non-policy characteristics, which allows for stable outcomes in electoral competition (pure strategy equilibria).

The rest of the paper is organized as follows. In the next section we present the model, in section 3 we analyze the model and provide equilibrium characterization results, in section 4 we analyze the variation of the model in which voters' valuations of candidates' non-policy characteristics are homogeneous and private information and in section 5 we conclude with some final discussion and remarks.

2 The model

The policy space is the $[0, 1]$ interval. There are two candidates, A and B , and each candidate's objective is to maximize his probability of winning the election. There are n voters, an odd and finite number.

Voters have a utility function with two components: a policy component and a candidate image component. The policy component is characterized by an ideal point in the policy space $x_i \in [0, 1]$, with utility of alternatives in the policy space a quadratic function of the distance between the ideal point and the location of the policy. The image component is captured by a constant $d_i \in \{-d, d\}$ such that $d > 0$, that is added to the utility a voter gets if candidate A wins the election.²

Let x denote the policy position chosen by candidate A and let y denote the policy position chosen by candidate B . Then, the utility that a voter with preference parameters (x_i, d_i) obtains if A wins the election is given by $U_i(x) = d_i - (x_i - x)^2$ and his utility if candidate B wins is given by $U_i(y) = -(x_i - y)^2$.

Both the ideal point of the voter and his valuation of the candidates non-policy characteristics are the voter's private information. We assume that both candidates have the same beliefs about voters' preferences that are common knowledge. Candidates' beliefs are given as follows: a) the ideal point of each voter is represented by an i.i.d. draw from a uniform distribution whose support is $[0, 1]$ and b) the image component of each voter is represented by an i.i.d. draw from a Bernoulli distribution with support $\{-d, d\}$. Let $\rho = \Pr(d_i = d)$ denote the probability that a voter values the non-policy characteristics of candidate A more than the ones of candidate B . Without loss of generality we assume that $\rho \in [\frac{1}{2}, 1]$.

The game takes place in two stages. In the first stage, candidates simultaneously choose policy positions in $[0, 1]$. In the second stage, voters vote for the candidate whose election would give them the

²We restrict attention to $d > 0$ as when $d = 0$ our model corresponds to the standard Downsian model: the unique equilibrium of the game is both candidates to choose the ideal policy of the median voter.

highest utility. In case of indifference, a voter is assumed to vote for each candidate with probability equal to $\frac{1}{2}$.

Since the behavior of the voters is unambiguous in this model, we define an equilibrium of the game only in terms of the location strategies of the two candidates in the first stage of the game. A pure strategy equilibrium is a pair of candidate locations (x, y) such that both candidates are maximizing the probability of winning given the choices of the other candidate. A mixed strategy equilibrium is a pair of probability distributions (σ^A, σ^B) over $[0, 1]$ such that there is no mixed strategy for A that guarantees higher probability of winning than σ^A , given σ^B , and there is no mixed strategy for B that guarantees higher probability of winning than σ^B , given σ^A .

Notice that in this setup all voters with $d_i - (x_i - x)^2 > -(x_i - y)^2$ prefer to vote for candidate A . Therefore, if $x < y$, we have that all voters with an ideal point $x_i < \frac{x+y}{2} + \frac{d_i}{2(y-x)} = \dot{x}(x, y, d_i)$ prefer to vote for candidate A . Since the ideal point of each voter is an i.i.d. draw from a uniform distribution and the non-policy component d_i is an i.i.d. draw from a Bernoulli distribution, the probability that a voter votes for candidate A is given by $p(x, y, \rho) = \rho \min \{\dot{x}(x, y, d), 1\} + (1 - \rho) \max \{0, \dot{x}(x, y, -d)\}$ and the probability that a voter votes for candidate B is given by $q(x, y, \rho) = 1 - p(x, y, \rho)$.

Similarly if $x > y$ we have that the probability that a voter votes for candidate A is given by $p(x, y, \rho) = \rho(1 - \max \{0, \dot{x}(x, y, d)\}) + (1 - \rho)(1 - \min \{\dot{x}(x, y, -d), 1\})$ and the probability that a voter votes for candidate B is given by $q(x, y, \rho) = 1 - p(x, y, \rho)$. Obviously, when $x = y$, we have that $p(x, y, \rho) = \rho$ and $q(x, y, \rho) = 1 - \rho$ because only the value of the image component will determine each voter's behavior.

Since voters' ideal points and candidates' images are independent random draws, the probability with which a candidate wins the election is given by the probability that he obtains the votes of at least a majority of the voters. Because each voter will vote for candidate A with probability $p(x, y, \rho)$ the probability with which candidate A is elected may be computed by the sum of the Bernoulli distributions corresponding to at least a majority of successes over n trials, that is,

$$P_n(x, y, \rho) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} p(x, y, \rho)^k (1 - p(x, y, \rho))^{n-k}.$$

Similarly we could also show that the probability with which B wins the election is given by

$$Q_n(x, y, \rho) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} q(x, y, \rho)^k (1 - q(x, y, \rho))^{n-k} = 1 - P_n(x, y, \rho).$$

Observe that $p(x, y, \rho)$ and $q(x, y, \rho)$ are continuous functions of $x \in [0, 1]$, $y \in [0, 1]$ and therefore $P_n(x, y, \rho)$ and $Q_n(x, y, \rho)$ are continuous functions of $x \in [0, 1]$, $y \in [0, 1]$. This guarantees that our

game admits at least one Nash equilibrium (possibly in mixed strategies) for any parameter values (see Glicksberg, 1952). Finally, notice that $P_n(x, y, \rho)$ is a strictly increasing function of $p(x, y, \rho)$ and similarly that $Q_n(x, y, \rho)$ is a strictly increasing function of $q(x, y, \rho)$.

3 Results

First we analyze the model for small values of $\rho \in [\frac{1}{2}, 1]$. In this case we find that, when ρ is small enough relative to d , there exist equilibria in pure strategies and in all of them candidates choose moderate strategies.

Proposition 1 *For all $d > 0$ and all $n \geq 1$:*

If $\rho \in [\frac{1}{2}, \frac{1}{2} + \frac{d}{2})$ any pure strategy profile $(x, y) \in [\frac{1}{2} - \frac{d}{2}, \frac{1}{2} + \frac{d}{2}]^2$ is a Nash equilibrium of the game.

If $\rho \in [\frac{1}{2} + \frac{d}{2}, \frac{1}{2} + 2d)$ any pure strategy profile $(x, y) \in [\rho - \sqrt{d(2\rho - 1)}, 1 - \rho + \sqrt{d(2\rho - 1)}] \times [\frac{1}{2} - \frac{d}{2}, \frac{1}{2} + \frac{d}{2}]$ is a Nash equilibrium of the game.

In all these equilibria $P_n(x, y, \rho) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \rho^k (1 - \rho)^{n-k}$ and $Q_n(x, y, \rho) = 1 - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \rho^k (1 - \rho)^{n-k}$.

There are no other pure strategy equilibria.

The multiple pure strategy equilibria obtained are all concentrated in a neighborhood of $\frac{1}{2}$. As ρ increases from $\frac{1}{2}$ to $\frac{1}{2} + \frac{d}{2}$ the equilibrium set of strategies for the each candidate remains unchanged, while as ρ increases from $\frac{1}{2} + \frac{d}{2}$ to $\frac{1}{2} + 2d$ the set of equilibrium strategies of A shrinks around the centre (see figures 1 and 3) while the set of equilibrium strategies of B remains the same (see figures 2 and 4). The size of the neighborhood that contains the equilibrium strategies depends on the difference between candidates' qualities (d) and on the probability with which candidate A enjoys the advantage (ρ) (see figures 1 to 4). On the one hand, the larger the difference between the candidates' qualities (d), the larger the set of policies that can be chosen in these equilibria. On the other hand, the larger the probability with which candidate A enjoys the advantage (ρ), the smaller the set of policies that can be chosen in these equilibria. This set becomes smaller because when ρ increases, candidate B becomes less likely to enjoy the advantage and therefore has more incentives to deviate to more extreme policies in order to differentiate from candidate A . This is avoided when candidate A becomes more moderate as it induces candidate B to keep choosing moderate policies. As the probability with which candidate A enjoys the advantage (ρ) increases, the set of pure strategies that candidate A chooses in equilibrium converges to a singleton equal to $\frac{1}{2}$. As ρ increases further the equilibrium can only exist in mixed strategies, because the expected share of voters who prefer the non-policy characteristics of candidate B decreases very much and

thus it becomes more profitable for candidate B to politicize the elections; to differentiate from candidate A to a significant extent.

[Insert figures 1, 2, 3 and 4 about here]

Notice that in all the pure strategy equilibria the vote is completely depoliticized - the equilibrium payoffs of both candidates are independent of the policy platforms they propose and depend only on ρ . This indicates that when no candidate is expected to enjoy a large enough advantage, the elections' outcome will be determined by the preferences of the voters only on one dimension - on how they evaluate the non-policy characteristics of candidates. The distribution of ideal policies will play absolutely no role on the determination of the political outcome which is, to the authors' eyes at least, the strongest implication of the above result.

As far as mixed equilibria of this case are concerned we observe the following. Given multiplicity of pure equilibria and the zero-sum nature of the game it is straightforward that a pair of probability distributions on the set of equilibrium strategies characterized by the above proposition forms a mixed strategy equilibrium. Therefore, we also have a multiplicity of mixed strategy equilibria but all equilibria, pure and mixed, yield the same moderate policy outcomes.

In order to complete the analysis we should analyze the equilibrium outcome for larger values of ρ - for cases in which candidate A is expected to be the favored candidate of the vast majority of voters. Observe that in the above case as the value of ρ approaches $\frac{1}{2} + 2d$ the equilibrium strategies of candidate A converge to $\frac{1}{2}$, thus in the limit candidate A is left with a unique equilibrium strategy equal to $\frac{1}{2}$. However all the equilibrium strategies for candidate B survive as best responses when the value of ρ grows. This seems to suggest that for larger values of ρ candidate A can continue to use a pure strategy equal to $\frac{1}{2}$ while candidate B will have to mix among strategies that are around $\frac{1}{2}$. This is what we show next. First we show that if candidate A chooses policy $\frac{1}{2}$ with probability one, a best response of candidate B is a mixed strategy with support two strategies equidistantly away from $\frac{1}{2}$. Notice that if $d > \frac{1}{4}$ then we have that $\frac{1}{2} + 2d > 1$ and hence it is always the case that $\rho < \frac{1}{2} + 2d$. Therefore, for such large values of d the premises of the previous proposition directly apply to all possible values of ρ . This is why here we focus on values of d that are smaller or equal than $1/4$.

Lemma 1 *For all $n \geq 1$ and for all $0 < d \leq \frac{1}{4}$ we have that when $\rho \in [\frac{1}{2} + 2d, 1]$ then $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$ with probability 50%) is a best response to $\tilde{\sigma}^A = \frac{1}{2}$.*

As we predicted, this lemma shows that when ρ is large enough and candidate A chooses policy $\frac{1}{2}$ then candidate B wants to use a mixed strategy with support around $\frac{1}{2}$. Next we show that if candidate B chooses the mixed strategy proposed above then the best response for candidate A is to choose $\frac{1}{2}$ with probability one.

Lemma 2 *For $0 < d \leq \frac{1}{4}$ and $\rho \in [\frac{1}{2} + 2d, 1]$ we have that $\tilde{\sigma}^A = \frac{1}{2}$ is a best response to $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$ with probability 50%) if n is sufficiently large (in specific, if $n \geq \frac{1}{4d(2\rho - 1)}$).*

From the combination of the two previous lemmata we essentially obtain that for large values of ρ , the mixed strategy profile $\tilde{\sigma}^A = \frac{1}{2}$ and $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$ with probability 50%) is a Nash equilibrium of the game if n is large enough ($n \geq \frac{1}{4d(2\rho - 1)}$).

Since the threshold of n that determines the existence of the describes mixed strategy equilibrium, $\frac{1}{4d(2\rho - 1)}$, is decreasing in ρ and this equilibrium only exists for $\rho \in [\frac{1}{2} + 2d, 1]$ it trivially follows that a sufficient condition for this equilibrium to exist for any $\rho \in [\frac{1}{2} + 2d, 1]$ is that $n \geq \frac{1}{4d(2(\frac{1}{2} + 2d) - 1)} = \frac{1}{16d^2} = (\frac{1}{4d})^2$ which coincides with the square of the threshold for the population size that we had for the complete information model³ (that is, of the case in which $\rho = 1$). As the value of ρ tends to one, the strategies of candidate B approach $\frac{1}{2} - \sqrt{d}$ and $\frac{1}{2} + \sqrt{d}$ and in the limit we obtain the ones described for the complete information model: $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d}$ with probability 50%). See figure 5. Finally, we prove that if this equilibrium exists then it is unique.

[Insert figure 5 about here]

Proposition 2 *For $0 < d \leq \frac{1}{4}$, $\rho \in [\frac{1}{2} + 2d, 1]$ and $n \geq \frac{1}{4d(2\rho - 1)}$ we have that $\tilde{\sigma}^A = \frac{1}{2}$ and $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$ with probability 50%) is the unique Nash equilibrium.*

Hence we have shown that in this model we find multiple equilibria in pure strategies for all n when ρ is small, and we find a unique equilibrium in mixed strategies for large values of ρ and a sufficiently large number of voters. We have that existence of pure strategy equilibria is more likely for smaller values of d and for smaller values of ρ , that is, when the advantage plays a smaller role and both candidates are more alike. The differentiation exhibited in the policy outcomes depends also on the two parameters that determine the advantage in this model. In the mixed equilibrium case, that is when ρ is large enough with respect to d , we have that larger parameter values of the advantage imply larger policy differentiation

³See Aragonès and Xefteris (2012).

in equilibrium. However, in the pure strategy equilibrium case, that is when ρ is small enough with respect to d , we have that larger values of d imply larger policy differentiation in equilibrium and instead larger values of ρ imply smaller differentiation in equilibrium. Thus, in this case, we have that maximum equilibrium policy differentiation is not monotonic with respect to all the parameters that define the advantage. See figures 6 and 7.

[Insert figures 6 and 7 about here]

In these mixed strategy equilibria we have that the candidate A 's payoffs increase with d , ρ , and n . That is, increases in any parameter value that represents the advantage benefits candidate A and increasing the number of voters also increases candidate A 's equilibrium payoffs. What is more important is that these equilibria lead to politicized elections - both dimensions are relevant for a voter's choice. This is because in these equilibria the disadvantaged candidate will locate quite far from the advantaged candidate and hence even if a voter prefers the non-policy characteristics of the advantaged candidate it may still be the case that he will vote for the disadvantaged one because he might think that the policy platform the disadvantaged candidate proposes is so much better than the one proposed by the advantaged one. This implies that when elections have a clear favorite, voting will not be only about non-policy characteristics but about policies as well.

4 Homogeneous and private valuations

In this part of the paper we investigate a variation of the model in which all voters favor the same candidate but candidates do not know who is the favored one. Apart from a robustness check to the model presented above, this variation of the model is of independent interest because it captures the effect of potential candidate endorsement by well thought off institutions or individuals on electoral competition dynamics. Consider the example in which the main policy issue is redistribution policy (determination of a flat tax rate for example) and that voters care not only about redistribution but about the moral values of the candidates as well. In this framework when an individual or organization, which is well thought off by the voters (for example a local religious leader or a church), announces which one of the two candidates he believes to be more moral than the other, then this candidate naturally gains a non-policy advantage. If it is the case that candidates, at the time in which they select platforms, are uncertain about which one of them will be endorsed by the well-thought-off third party, then their problem is well described by a model in which all voters value one candidate more (homogeneous valuations) than the other but candidates are uncertain about which of them is the favored one (candidate valuations are voters' private

information). These are precisely the assumptions of the variation of the original model that we intend to analyze in this section.

For economy of space we do not replicate all the assumptions: anything that is not explicitly defined here is exactly as in the main model. Candidates' beliefs are given as follows: a) the ideal point of each voter is represented by an i.i.d. draw from a uniform distribution whose support is $[0, 1]$ and b) the image component of each voter is represented by a unique (*common to all voters*) draw from a Bernoulli distribution with support $\{-d, d\}$. Let $\hat{\rho} = \Pr(\hat{d} = d)$ denote the probability with which candidate A enjoys the advantage. That is, candidate A appears superior to candidate B ($\hat{d} = d$) to all voters with probability $\hat{\rho}$ and candidate B will be superior to candidate A ($\hat{d} = -d$) for all voters with probability $1 - \hat{\rho}$. Without loss of generality we assume here too that $\hat{\rho} \in [\frac{1}{2}, 1]$. Notice that in this case voters' ideal points are generated by independent random draws but candidates' images are generated by a common draw.

The game takes place in two stages. In the first stage, candidates simultaneously choose policy positions in $[0, 1]$. In the second stage, voters vote for the candidate whose election would give them the highest utility. In case of indifference, a voter is assumed to vote for each candidate with probability equal to $1/2$.

Since the behavior of the voters is unambiguous in this model too, we define an equilibrium of the game again only in terms of the location strategies of the two candidates in the first stage of the game. A pure strategy equilibrium is a pair of candidate locations (\hat{x}, \hat{y}) such that both candidates are maximizing the probability of winning given the choices of the other candidate. A mixed strategy equilibrium is a pair of probability distributions $(\hat{\sigma}^A, \hat{\sigma}^B)$ over $[0, 1]$ such that there is no mixed strategy for A that guarantees higher probability of winning than $\hat{\sigma}^A$, given $\hat{\sigma}^B$, and there is no mixed strategy for B that guarantees higher probability of winning than $\hat{\sigma}^B$, given $\hat{\sigma}^A$.

In this set up all voters with $\hat{d} - (\hat{x}_i - \hat{x})^2 > -(\hat{x}_i - \hat{y})^2$ prefer to vote for candidate A . Therefore, if $\hat{x} < \hat{y}$ all voters with an ideal point $\hat{x}_i < \frac{\hat{x} + \hat{y}}{2} + \frac{\hat{d}}{2(\hat{y} - \hat{x})} = \bar{x}(\hat{x}, \hat{y}, \hat{d})$ prefer to vote for candidate A . For simplicity of exposition of the coming arguments we define $\bar{x}(\hat{x}, \hat{y}, \hat{d}) = \max\{0, \min\{\bar{x}(\hat{x}, \hat{y}, \hat{d}), 1\}\}$.

Let $\hat{p}(\hat{x}, \hat{y}, \hat{d})$ denote the probability that a voter votes for candidate A conditional on the realization of the common image component, \hat{d} , and let $\hat{q}(\hat{x}, \hat{y}, \hat{d})$ the probability that a voter votes for candidate B conditional on the realization of the common image component, \hat{d} .

Since the ideal point of each voter is an i.i.d. draw from a continuous distribution and the non-policy component \hat{d} is a common draw from a Bernoulli distribution, the probability that a voter votes for candidate A conditional on the realization of the common image component, \hat{d} , should be given by $\hat{p}(\hat{x}, \hat{y}, \hat{d}) = \bar{x}(\hat{x}, \hat{y}, \hat{d})$ and the probability that a voter votes for candidate B conditional on the realization

of the common image component, \hat{d} , should be given by $\hat{q}(\hat{x}, \hat{y}, \hat{d}) = 1 - \hat{p}(\hat{x}, \hat{y}, \hat{d})$.

Similarly if $\hat{x} > \hat{y}$ we have that the probability that a voter votes for candidate A conditional on the realization of the common image component, \hat{d} , should be given by $\hat{p}(\hat{x}, \hat{y}, \hat{d}) = 1 - \bar{x}(\hat{x}, \hat{y}, \hat{d})$ and the probability that a voter votes for candidate B conditional on the realization of the common image component, \hat{d} , should be given by $\hat{q}(\hat{x}, \hat{y}, \hat{d}) = 1 - \hat{p}(\hat{x}, \hat{y}, \hat{d})$. Obviously, when $\hat{x} = \hat{y}$, we have that $\hat{p}(\hat{x}, \hat{y}, \hat{d}) = 1$ if $\hat{d} = d$, $\hat{p}(\hat{x}, \hat{y}, \hat{d}) = 0$ if $\hat{d} = -d$ and $\hat{q}(\hat{x}, \hat{y}, \hat{d}) = 1 - \hat{p}(\hat{x}, \hat{y}, \hat{d})$ because only the value of the image component will determine all voters behavior.

The probability with which a candidate wins the election is given by the probability that he obtains the votes of at least a majority of the voters. Because each voter will vote for candidate A with probability $\hat{p}(\hat{x}, \hat{y}, d)$ if $\hat{d} = d$ and with probability $\hat{p}(\hat{x}, \hat{y}, -d)$ if $\hat{d} = -d$, the probability with which candidate A is elected may be computed by the sum of the Bernoulli distributions corresponding to at least a majority of successes over n trials in each one of the possible states of the world defined according to the sign of the advantage, that is,

$$\hat{P}_n(\hat{x}, \hat{y}, \hat{\rho}) = \hat{\rho} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{p}(\hat{x}, \hat{y}, d)^k (1 - \hat{p}(\hat{x}, \hat{y}, d))^{n-k} + (1 - \hat{\rho}) \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{p}(\hat{x}, \hat{y}, -d)^k (1 - \hat{p}(\hat{x}, \hat{y}, -d))^{n-k}.$$

Similarly we also have that the probability with which candidate B wins the election is given by

$$\begin{aligned} \hat{Q}_n(\hat{x}, \hat{y}, \hat{\rho}) &= \hat{\rho} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\hat{x}, \hat{y}, d)^k (1 - \hat{q}(\hat{x}, \hat{y}, d))^{n-k} + (1 - \hat{\rho}) \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\hat{x}, \hat{y}, -d)^k (1 - \hat{q}(\hat{x}, \hat{y}, -d))^{n-k} = \\ &= 1 - \hat{P}_n(\hat{x}, \hat{y}, \hat{\rho}). \end{aligned}$$

Observe that $\hat{p}(\hat{x}, \hat{y}, \hat{d})$ and $\hat{q}(\hat{x}, \hat{y}, \hat{d})$ are continuous functions of $\hat{x} \in [0, 1]$ and $\hat{y} \in [0, 1]$ and therefore $\hat{P}_n(\hat{x}, \hat{y}, \hat{\rho})$ and $\hat{Q}_n(\hat{x}, \hat{y}, \hat{\rho})$ are continuous functions of $\hat{x} \in [0, 1]$, $\hat{y} \in [0, 1]$ as well. Again, Glicksberg's (1952) theorem applies and a Nash equilibrium is guaranteed to exist.

Since n is assumed to be odd there exists a unique median voter. Because candidates are assumed to maximize their probability of winning and because all voters value equally candidates' non-policy characteristics, candidates are in fact maximizing the probability that the median voter votes for them.⁴

In our case the distribution of the median for a fixed value of \hat{d} corresponds to the distribution of a sample of size n drawn from a uniform distribution, and in turn it coincides with the distribution of the $\frac{n+1}{2}$ th order statistic of such a sample which is distributed according to a Beta distribution with parameters

⁴Groseclose (2007) shows that in such one and a half dimensional domains the alternative preferred by the median voter is the majority winner.

$a = b = \frac{n+1}{2}$, that is $\beta_n(\hat{x}, \hat{y}, \hat{d}) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{p}(\hat{x}, \hat{y}, \hat{d})^k (1 - \hat{p}(\hat{x}, \hat{y}, \hat{d}))^{n-k}$. The density function of such a Beta distribution is unimodal and symmetric around $\frac{1}{2}$ and its variance $Var\left(\text{Beta}\left(\frac{n+1}{2}, \frac{n+1}{2}\right)\right) = \frac{1}{4n+8}$ decreases with n , that is, with a larger number of voters, the variance becomes smaller, and the probability that the ideal point of the median voter is close to $\frac{1}{2}$ increases.

This implies that the probability that the median votes for candidate A is given by $\hat{P}_n(\hat{x}, \hat{y}, \hat{\rho}) = \rho\beta_n(\hat{x}, \hat{y}, d) + (1 - \hat{\rho})\beta_n(\hat{x}, \hat{y}, -d)$, that is a convex combination of two Beta distributions, and this is exactly the payoff function that we consider for candidate A .

Observe that, as it happens in the model with a complete information advantage, when the difference between the candidates' quality is very large ($d \geq \frac{1}{4}$), the policy choices have no effect in determining the candidates' best responses. In this case many pure strategy equilibria exist. If A locates at $\frac{1}{2}$, then B gets a payoff of $1 - \hat{\rho}$ independently of where he locates and when B locates at $\frac{1}{2}$, then A gets a payoff of $\hat{\rho}$ independently of where he locates. Thus for large values of d the model is not very interesting strategy-wise. Therefore, we restrict our attention to values of d that are smaller than $\frac{1}{4}$.

Proposition 3 *If $0 < d < \frac{1}{4}$ and $\frac{1}{2} \leq \hat{\rho} < 1$ there exists $\tilde{n}(\hat{\rho}, d)$ such that for $n > \tilde{n}(\hat{\rho}, d)$ all the pure strategy profiles (\hat{x}, \hat{y}) such that $|\hat{x} - \frac{1}{2}| < \varepsilon_{\hat{x}}$ and $|\hat{y} - \frac{1}{2}| < \varepsilon_{\hat{y}}$ for $\varepsilon_{\hat{x}}, \varepsilon_{\hat{y}} < \varepsilon(\hat{\rho}, d)$ are Nash equilibria of the game. In all these equilibria $\hat{P}_n(\hat{x}, \hat{y}, \hat{\rho}) = \hat{\rho}$ and $\hat{Q}_n(\hat{x}, \hat{y}, \hat{\rho}) = 1 - \hat{\rho}$. For $\hat{\rho} = 1$ no pure strategy equilibrium exists for any $0 < d < \frac{1}{4}$ and any $n > 0$.*

This proposition shows that moderate convergence to the expected median's ideal point in pure strategies by both parties is an equilibrium of this variation of the model for all values of $\hat{\rho}$ as long as n is large enough. Notice that here again we obtain a multiplicity of equilibria in pure strategies that leads to the same candidates' payoffs. This implies that any combination of the candidates' strategies characterized by this proposition forms a mixed strategy equilibrium, therefore, we also have a multiplicity of mixed strategy equilibria.

The variance of the distribution of the ideal point of the median voter, which in this model is parametrized by n seems to be an important parameter here. Since large n implies large probability that the median voter's ideal point is close to $\frac{1}{2}$, then, it should be expected that as n increases both parties tend to prefer policies that are close to $\frac{1}{2}$ and subsequently that these equilibria lead to a depoliticized vote (a candidate is elected if and only if he is his non-policy characteristics are considered to be better than those of the other candidate). Observe that this large enough n which guarantees existence of pure strategy equilibria can be as small as one. To see this notice that this variation of our model is identical to the original game for the case in which there exists a unique voter. Hence, when $n = 1$, all equilibria

of the original game characterized by Proposition 1 are equilibria of the present variation of the game too. In this case though, when pure equilibria do not exist a full characterization of mixed equilibria is intractable. This is so because the best response of B to a A playing a fixed pure strategy depends on n too - in the original model the best response of B to a A playing a fixed pure strategy depends only on d and on ρ - and this complicates formal analysis to several degrees of magnitude. Despite this one can still describe qualitative features of such mixed equilibria: when only mixed equilibria exist a) A behaves, in expected terms, more moderately than B and b) A is elected with larger probability than B .

5 Concluding remarks

Our analysis shows that when we combine an advantage for one of the candidates with incomplete information of the candidates about the voters preferences over the non-policy candidates' characteristics, we obtain existence of pure strategy equilibria for a large range of parameter values and the possibility of diverging pure strategies in equilibrium. These features are driven by the incomplete information of candidates with respect to the advantage, because they do not appear at all in the models with complete information advantage. Indeed, when the private information of voters with respect to their valuation of the candidates' quality disappears, then the equilibrium in both models coincides with the equilibrium of the original model with a deterministic advantage.

The reason is that when we introduce incomplete information the advantage acquires a new role. In fact, in our model the advantage is described with two different parameters. One that refers to the difference between the candidates' qualities, that is already present in the models of complete information. And a different parameter that refers to the probability with which each candidate is expected to enjoy the advantage, which is specific of our incomplete information advantage model.

We have that these two parameters play different roles: for larger values of the difference between the candidates' qualities existence of pure strategy equilibrium is more likely, however for larger values of the probability that candidate A enjoys the advantage existence of only mixed strategy equilibria is more likely. In addition, our equilibrium outcomes show that the degree of policy divergence increases in all cases with the size of the difference between the candidates' quality, following the intuition that was already known from the existing models of complete information about the advantage. However this is not the case when the other advantage parameter increases. When it becomes more and more likely that candidate A is the advantaged one, we can obtain more or less divergence in equilibrium depending on whether the size of this probability is large or small. Thus we obtain that the equilibrium policy divergence is not monotonic with respect to the size of the advantage.

The introduction of private information for the voters regarding their perception of the non-policy characteristics of the candidates has some general implications such as:

(i) Pure strategy equilibria are only ruled out when voters are sufficiently homogeneous as far as how they value the non-policy characteristics of a candidate.

(ii) Pure strategy equilibria may be diverging.

(iii) Increasing the asymmetry of the model (ρ) makes mixed strategy equilibria more likely, while increasing the difference between the candidate's quality (d) makes pure strategy equilibria more likely.

(iv) When voters are sufficiently heterogeneous as far as how they value the non-policy characteristics of a candidate then the vote is depoliticized and when they are sufficiently homogeneous in that respect then the vote is politicized.

Finally, since we have modeled preferences on non-policy components as an additive constant our analysis fits in the class of differentiated candidates models analyzed by Dziubinski and Roy (2011) and Krasa and Polborn (2012), among others. Our paper contributes to this literature by providing a characterization of global Nash equilibria in a family of setups where most results are based on computations of local Nash equilibria and sufficient conditions for existence of Nash equilibria.

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6 Appendix

6.1 Proofs

Proof of Proposition 1. We assume that $n = 1$. First we will argue that $(x, y) = (\frac{1}{2}, \frac{1}{2})$ is an equilibrium of the game when $\rho \in [\frac{1}{2}, \frac{1}{2} + 2d]$. If $y = \frac{1}{2}$ it is trivial to check that $P_1(x, \frac{1}{2}, \rho) \leq \rho$ for every $(x, \rho) \in [0, 1] \times [\frac{1}{2}, 1]$ and that $P_1(\frac{1}{2}, \frac{1}{2}, \rho) = \rho$; $x = \frac{1}{2}$ is a best response to B playing $y = 1$ (detailed arguments which back up this claim are presented in Step 1 of the present proof). If $x = \frac{1}{2}$ then, by the detailed arguments which are available in the proof of Lemma 1, it is clear that $y = \frac{1}{2}$ is a best response of B if and only if $\rho \in [\frac{1}{2}, \frac{1}{2} + 2d]$. Hence, $(x, y) = (\frac{1}{2}, \frac{1}{2})$ is an equilibrium of the game when $\rho \in [\frac{1}{2}, \frac{1}{2} + 2d]$. The payoff of A in this equilibrium is ρ and the payoff B is $1 - \rho$. The zero-sum nature of the game dictates that if other equilibria exist for these parameter values then the payoffs of the candidates in these equilibria will be the same with the one we have identified here.

Having in mind that our game is zero-sum we characterize the set of pure equilibria (or the sets of minimaximizer strategies for both players) in two steps: first we identify the set of pure strategy minimaximizers of player B and then we do the same for player A .

Step 1.

We know that the equilibrium payoff of A is ρ . So if y is a minimaximizer strategy of B then it should be the case that for any $x \in [0, 1]$ we have that $P_1(x, y, \rho) \leq \rho$.

Assume that B is playing some $y \geq \frac{1}{2}$ and let us check whether it is a minimaximizer strategy or not.

First we note that for any $x = y$ we have that $P_1(x, y, \rho) = \rho$.

We define $o_A(y) = 1 - \sqrt{1 + d - 2y + y^2}$ that represents the most leftist policy that candidate A can choose with $x \leq y$ that allows him to win with probability 1 when he is the most preferred candidate, that is, even a voter with ideal point $x_i = 1$ would prefer to vote for candidate A .

Similarly we define $o_B(y) = \sqrt{-d + y^2}$ that represents the most leftist policy that candidate A can choose with $x \leq y$ that makes him lose with probability 1 when he is the least preferred candidate, that is, even a voter with ideal point $x_i = 0$ would prefer to vote for candidate B .

First, by the means of standard algebraic manipulations, we get that $o_A(y) > o_B(y)$ if and only if $\frac{1+d}{2} > y$. Then we notice the following: if $o_A(y) < o_B(y)$ (or else if $\frac{1+d}{2} < y$) it is true that $\frac{\partial P_1(x, y, \rho)}{\partial x} = \frac{(-1+\rho)(d-(x-y)^2)}{2(x-y)^2} < 0$ for all $x \in (o_A(y), o_B(y))$ and $P_1(o_B(y), y, \rho) = \rho$. That is if A locates arbitrarily to the left of $o_B(y)$ he gets a payoff strictly larger than ρ ; thus y cannot be a minimaximizer strategy of player B .

On the other hand if $o_A(y) \geq o_B(y)$, that is, if $\frac{1}{2} \leq y \leq \frac{1+d}{2}$ then it is the case that $\frac{\partial P_1(x, y, \rho)}{\partial x} = \frac{1}{2}\rho \left(1 + \frac{d}{(x-y)^2}\right) > 0$ for all $x \in (o_B(y), o_A(y))$ therefore for all $\frac{1}{2} < y < \frac{1+d}{2}$ candidate A prefers $o_A(y)$ to

any location in $(o_B(y), o_A(y))$ (notice also that in this case $P_1(x, y, \rho) = \rho$ for every $x \in [o_A(y), y]$).

Finally, we simply observe that for every $\frac{1}{2} \leq y \leq \frac{1+d}{2}$ it is true that $\frac{\partial P_1(x, \hat{y}, \rho)}{\partial x} = \frac{1}{2} \left(1 + \frac{d(-1+2\rho)}{(x-y)^2} \right) > 0$ for $x \in [0, o_B(y))$ and hence when $\frac{1}{2} \leq y \leq \frac{1+d}{2}$ we have that $P_1(x, y, \rho) \leq \rho$ for any $x \in [0, y]$.

One can show with similar (actually easier) arguments that the same holds for any $x \in (y, 1]$.

We have just demonstrated that:

a) if $\frac{1}{2} \leq y \leq \frac{1+d}{2}$ player A cannot get a payoff larger than ρ independently of what strategy he chooses to play, that is, any $\frac{1}{2} \leq y \leq \frac{1+d}{2}$ is a minimaximizer for player B .

b) if $\frac{1+d}{2} < y$ then player A can get a payoff larger than ρ , that is, any $y > \frac{1+d}{2}$ is not a minimaximizer for B .

By symmetry one can show that every $y \in [\frac{1}{2} - \frac{d}{2}, \frac{1}{2}]$ is a minimaximizer of player B when $\rho \in [\frac{1}{2}, \frac{1}{2} + 2d)$ and $n = 1$.

Step 2.

We have shown in proposition 1 that the equilibrium payoff of B is $1 - \rho$. So if x is a minimaximizer strategy of A then it should be the case that for any $y \in [0, 1]$ we have that $Q_1(x, y, \rho) \leq 1 - \rho$.

Assume that A is playing some $x \geq \frac{1}{2}$ and let us check whether it is a minimaximizer strategy.

First we note that for any $x = y$ we have that $Q_1(x, y, \rho) = 1 - \rho$.

We define $e_B(x) = 1 - \sqrt{1 + d - 2x + x^2}$ that represents the most leftist policy that candidate B can choose with $y \leq x$ that allows him to win with probability 1 when he is the most preferred candidate, that is, even a voter with ideal point $x_i = 1$ would prefer to vote for candidate B .

Similarly we define $e_A(x) = \sqrt{-d + x^2}$ that represents the most leftist policy that candidate B can choose with $y \leq x$ that makes him lose with probability 1 when he is the least preferred candidate, that is, even a voter with ideal point $x_i = 0$ would prefer to vote for candidate A .

Using the argument from the previous step, we can show that $e_A(x) \leq e_B(x)$ if and only if $\frac{1+d}{2} > x$.

Notice that here we have that $\frac{\partial Q_1(x, y, \rho)}{\partial y} > 0$ for every $y \in (0, e_A(x))$ only when $x - \sqrt{d(2\rho - 1)} > e_A(x)$.

Otherwise, when $x - \sqrt{d(2\rho - 1)} < e_A(x)$, then for $y \in [0, e_A(x)]$ we have that $y = x - \sqrt{d(2\rho - 1)}$ maximizes $Q_1(x, y, \rho)$. So, when $x - \sqrt{d(2\rho - 1)} < e_A(x)$, $x \geq \frac{1}{2}$ will not be a pure strategy minimaximizer of player A if $Q_1(x, x - \sqrt{d(2\rho - 1)}, \rho) > 1 - \rho$.

By simple algebraic manipulations we get that $x - \sqrt{d(2\rho - 1)} < e_A(x)$ and $Q_1(x, x - \sqrt{d(2\rho - 1)}, \rho) > 1 - \rho$ are all simultaneously true if and only if $x > 1 - \rho + \sqrt{d(2\rho - 1)}$ and $\frac{1+d}{2} < \rho < \frac{1}{2} + 2d$. In other words there exists $\hat{y} \in [0, x]$ such that $Q_1(x, \hat{y}, \rho) > 1 - \rho$ if and only if $x > 1 - \rho + \sqrt{d(2\rho - 1)}$ and $\frac{1+d}{2} < \rho < \frac{1}{2} + 2d$.

Hence, when $\rho \leq \frac{1+d}{2}$ it is the case that for every $y \in [0, x]$ we have $Q_1(x, y, \rho) \leq 1 - \rho$ as long as $x \in [\frac{1}{2}, \frac{1}{2} + \frac{d}{2}]$.

While for $\frac{1+d}{2} < \rho < \frac{1}{2} + 2d$ it is the case that for every $y \in [0, x]$ we have $Q_1(x, y, \rho) \leq 1 - \rho$ as long as $x \in [\frac{1}{2}, \min\{\frac{1}{2} + \frac{d}{2}, 1 - \rho + \sqrt{d(2\rho - 1)}\}]$.

Notice that $1 - \rho + \sqrt{d(2\rho - 1)} = \frac{1}{2} + \frac{d}{2}$ when $\rho = \frac{1+d}{2}$ and $1 - \rho + \sqrt{d(2\rho - 1)} < \frac{1}{2} + \frac{d}{2}$ when $\rho > \frac{1+d}{2}$.

One can show with similar (actually easier) arguments that these conditions on x guarantee that $Q_1(x, y, \rho) \leq 1 - \rho$ for all $y \in (x, 1]$ too as long as x satisfies the above conditions.

Therefore, any $x \in [\frac{1}{2} - \frac{d}{2}, \frac{1}{2} + \frac{d}{2}]$ is a minimaximizer of player A when $\rho \in [\frac{1}{2}, \frac{1}{2} + \frac{d}{2}]$ and any $x \in [\rho - \sqrt{d(2\rho - 1)}, 1 - \rho + \sqrt{d(2\rho - 1)}]$ is a minimaximizer of player A when $\rho \in [\frac{1}{2} + \frac{d}{2}, \frac{1}{2} + 2d]$.

Now if $n > 1$ we notice that it is true that $P_n(x, y, \rho) = G(P_1(x, y, \rho))$ where $G(x) = \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} x^k (1-x)^{n-k}$ which is strictly increasing $x \in (0, 1)$; $P_n(x, y, \rho)$ is a monotone transformation of $P_1(x, y, \rho)$. Therefore, if for some $y \in [0, 1]$ we have that x^* maximizes $P_1(x, y, \rho)$ it must also be the case that it maximizes $P_n(x, y, \rho)$. In other words this model is such that if it has a pure strategy equilibrium for $n = 1$ then this equilibrium exists for any n . ■

Proof of Lemma 1. Consider that A locates at $x = \frac{1}{2}$. Then since $Q_n(x, y, \rho)$ is strictly increasing in $q(x, y, \rho)$ it follows that $Q_n(\frac{1}{2}, y, \rho)$ is maximized when $q(\frac{1}{2}, y, \rho)$ is maximized. As we know for $y < x$ we have that

$$q(x, y, \rho) = 1 - [\rho(1 - \max\{0, \dot{x}(x, y, d)\}) + (1 - \rho)(1 - \min\{\dot{x}(x, y, -d), 1\})]$$

and hence finding the values of y for which $q(\frac{1}{2}, y, \rho)$ takes its maximal value depends on understanding the functions $\dot{x}(\frac{1}{2}, y, d)$ and $\dot{x}(\frac{1}{2}, y, -d)$.

We have studied these functions to some extent already in the proof of Proposition 1. In particular we know that:

- a) if $y \in [1 - \frac{1}{2}\sqrt{1+4d}, \frac{1}{2}]$ then $\dot{x}(\frac{1}{2}, y, d) \leq 0$ and $\dot{x}(\frac{1}{2}, y, -d) \geq 1$,
- b) if $y \in [\sqrt{\frac{1}{4}-d}, 1 - \frac{1}{2}\sqrt{1+4d}]$ then $\dot{x}(\frac{1}{2}, y, d) \leq 0$ and $\dot{x}(\frac{1}{2}, y, -d) \in (0, 1)$
- c) if $y < \sqrt{\frac{1}{4}-d}$ then $\dot{x}(\frac{1}{2}, y, d) \in (0, 1)$ and $\dot{x}(\frac{1}{2}, y, -d) \in (0, 1)$.

Therefore, $q(\frac{1}{2}, y, \rho) = 1 - \rho$ when $y \in [1 - \frac{1}{2}\sqrt{1+4d}, \frac{1}{2}]$ and $q(\frac{1}{2}, y, \rho) < 1 - \rho$ when $y \in [\sqrt{\frac{1}{4}-d}, 1 - \frac{1}{2}\sqrt{1+4d}]$ while it may be larger or smaller than $1 - \rho$ for $y \in [0, \sqrt{\frac{1}{4}-d}]$.

Since for $y \in [0, \sqrt{\frac{1}{4}-d}]$ we have that $\dot{x}(\frac{1}{2}, y, d) \leq 0$ and $\dot{x}(\frac{1}{2}, y, -d) \in (0, 1)$ it follows that $q(\frac{1}{2}, y, \rho) = 1 - [\rho(1 - \dot{x}(\frac{1}{2}, y, d)) + (1 - \rho)(1 - \dot{x}(\frac{1}{2}, y, -d))]$ which is at least twice differentiable in y .

By solving the standard maximization problem

$$\max_y \{q(\frac{1}{2}, y, \rho) \text{ s.t. } y \in [0, \sqrt{\frac{1}{4}-d}]\}$$

we get the solution:

$y^* = \frac{1}{2} - \sqrt{d(2\rho - 1)} < \sqrt{\frac{1}{4} - d}$ if $\rho > \frac{1}{4} \left(-\sqrt{\frac{1-4d}{d^2}} + \frac{1}{d} \right)$ and

$y^* = \sqrt{\frac{1}{4} - d}$ if $\rho < \frac{1}{4} \left(-\sqrt{\frac{1-4d}{d^2}} + \frac{1}{d} \right)$.

We observe that:

a) $q(\frac{1}{2}, \sqrt{\frac{1}{4} - d}, \rho) < 1 - \rho$ and that

b) $q(\frac{1}{2}, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho) = \frac{1}{2} - \sqrt{d(2\rho - 1)}$.

Moreover it is easy to check that:

a) $\frac{1}{2} - \sqrt{d(2\rho - 1)} > 1 - \rho$ if and only if $\rho > \frac{1}{2} + 2d$ and

b) $\rho > \frac{1}{2} + 2d$ if and only if $\rho > \frac{1}{4} \left(-\sqrt{\frac{1-4d}{d^2}} + \frac{1}{d} \right)$.

To sum up, we have demonstrated that when $\rho > \frac{1}{2} + 2d$ the function $q(\frac{1}{2}, y, \rho)$ - and subsequently the function $Q_n(\frac{1}{2}, y, \rho)$ - admits a maximum only at $y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ and (by symmetry) at $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$. This completes the proof of this lemma. ■

Proof of Lemma 2. Consider that B is using the strategy $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$ with probability 50%). Then we need to show that $P_n(x, \tilde{\sigma}^B, \rho)$ admits a maximum at $x = \frac{1}{2}$.

Since $P_n(x, \tilde{\sigma}^B, \rho) = \frac{1}{2} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho)^k (1 - p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho))^{n-k} + \frac{1}{2} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho)^k (1 - p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho))^{n-k}$

is continuous and both $p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho)$ and $p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho)$ can be easily shown to be increasing in $x \in [0, \sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}]$ and decreasing in $x \in [1 - \sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}, 1]$

it follows that to prove this lemma we only need to study $P_n(x, \tilde{\sigma}^B, \rho)$ for $x \in [\sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}, 1 - \sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}]$.

We notice that for such values of x it is the case that $\dot{x}(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, d) \in (0, 1)$, $\dot{x}(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, -d) \in (0, 1)$,

$\dot{x}(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, d) \in (0, 1)$ and $\dot{x}(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, -d) \in (0, 1)$. Hence, $P_n(x, \tilde{\sigma}^B, \rho)$ is at least twice differentiable in $x \in [\sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}, 1 - \sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}]$.

We compute the derivative of $P_n(x, \tilde{\sigma}^B, \rho)$ with respect to x and we get

$$\frac{\partial P_n(x, \tilde{\sigma}^B, \rho)}{\partial x} = \frac{n}{2} \binom{n-1}{\frac{n-1}{2}} \left[\left[p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho) (1 - p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho)) \right]^{\frac{n-1}{2}} \frac{\partial p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho)}{\partial x} + \left[p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho) (1 - p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho)) \right]^{\frac{n-1}{2}} \frac{\partial p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho)}{\partial x} \right]$$

First we will prove that for large enough values of n we have that

$$\left[p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho) (1 - p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho)) \right]^{\frac{n-1}{2}} \frac{\partial p(x, \frac{1}{2} - \sqrt{d(2\rho - 1)}, \rho)}{\partial x} + \left[p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho) (1 - p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho)) \right]^{\frac{n-1}{2}} \frac{\partial p(x, \frac{1}{2} + \sqrt{d(2\rho - 1)}, \rho)}{\partial x} > 0$$

whenever $x \in [\sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho - 1)}}, \frac{1}{2}]$.

This holds if and only if

$$\left[\frac{p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)\right)}{p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)\right)} \right]^{\frac{n-1}{2}} \frac{\partial p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)}{\partial x} + \frac{\partial p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)}{\partial x} > 0$$

Notice that $\left[\frac{p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)\right)}{p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)\right)} \right]^{\frac{n-1}{2}}$ decreases with n and it tends to zero as n tends to infinite.

This is because for $x \in [\sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho-1)}}, \frac{1}{2}]$ we have that $p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) > p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) > \frac{1}{2}$ which implies that

$$p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)\right) < p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)\right) \text{ always holds, since it does not depend on } n.$$

Thus $\left[\frac{p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)\right)}{p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)\right)} \right]^{\frac{n-1}{2}} < 1$ and $\lim_{n \rightarrow \infty} \left[\frac{p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)\right)}{p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)\right)} \right]^{\frac{n-1}{2}} = 0$

We have that $\frac{\partial p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)}{\partial x} > 0$ for any $x \in [\sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho-1)}}, \frac{1}{2}]$ and hence we also have

that $\left[\frac{p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)\right)}{p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho) \left(1 - p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)\right)} \right]^{\frac{n-1}{2}} \frac{\partial p(x, \frac{1}{2} - \sqrt{d(2\rho-1)}, \rho)}{\partial x} + \frac{\partial p(x, \frac{1}{2} + \sqrt{d(2\rho-1)}, \rho)}{\partial x} > 0$ for large (but finite) values of n .

Similarly we can show that for $x \in (\frac{1}{2}, 1 - \sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho-1)}}]$ we have $\frac{\partial P_n(x, \tilde{\sigma}^B, \rho)}{\partial x} < 0$ which for sufficiently large n .

Finally we compute $\frac{\partial^2 P_n(x, \tilde{\sigma}^B, \rho)}{\partial x^2} \Big|_{x=\frac{1}{2}}$ and we get that it is smaller or equal to zero if and only if $n \geq \frac{2[\frac{1}{4} - (\sqrt{d(2\rho-1)})^2]}{2(\sqrt{d(2\rho-1)})\sqrt{d(2\rho-1)}} + 1 = \frac{1}{4d(2\rho-1)}$.

So when $\rho \in [\frac{1}{2} + 2d, 1]$, for sufficiently large (but finite) values of n we actually have that $x = \frac{1}{2}$ is the unique maximum of $P_n(x, \tilde{\sigma}^B, \rho)$. Moreover one can show with computational support that for all values of n for which $\frac{\partial^2 P_n(x, \tilde{\sigma}^B, \rho)}{\partial x^2} \Big|_{x=\frac{1}{2}} \leq 0$ it is the case that $\frac{\partial P_n(x, \tilde{\sigma}^B, \rho)}{\partial x} > 0$ if $x \in [\sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho-1)}}, \frac{1}{2}]$ and that $\frac{\partial P_n(x, \tilde{\sigma}^B, \rho)}{\partial x} < 0$ if $x \in (\frac{1}{2}, 1 - \sqrt{\frac{1}{4} + 2d\rho - \sqrt{d(2\rho-1)}}]$; for any $n \geq \frac{1}{4d(2\rho-1)}$ we have that $x = \frac{1}{2}$ is a best-response to $\tilde{\sigma}^B = (y = \frac{1}{2} - \sqrt{d(2\rho-1)})$ with probability 50% and $y = \frac{1}{2} + \sqrt{d(2\rho-1)}$ with probability 50%. ■

Proof of Proposition 2. Uniqueness is implied by the arguments used in the proofs of the previous lemmata. If B uses $\tilde{\sigma}^B$ then we have that $\tilde{\sigma}^A = \frac{1}{2}$ is the unique best response of A. That is, $\frac{1}{2}$ is the unique minimaximizer strategy of A or else the unique strategy that A may use in an equilibrium. If A uses $\tilde{\sigma}^A = \frac{1}{2}$ then we have shown that B has exactly two best responses, $y = \frac{1}{2} - \sqrt{d(2\rho-1)}$ and $y = \frac{1}{2} + \sqrt{d(2\rho-1)}$. But in an any mixture between them except for the even, A has a best response different than $\frac{1}{2}$ which gives him a larger payoff than him minimaximizer strategy. Hence, the even

mixture between $y = \frac{1}{2} - \sqrt{d(2\rho - 1)}$ and $y = \frac{1}{2} + \sqrt{d(2\rho - 1)}$ is the unique minimaximizer strategy of B . ■

Proof of Proposition 3. Consider that A is expected to locate at $\hat{x} = \frac{1}{2}$. Then B by locating at $\hat{y} = \frac{1}{2}$ gets elected with probability $1 - \hat{\rho}$. If B deviates to some $\hat{y} < \hat{x} = \frac{1}{2}$ then three things may occur.

Case 1. If B deviates to $\hat{y} \in [1 - \frac{1}{2}\sqrt{1+4d}, \frac{1}{2})$ then his payoff remains unaffected.

Notice that $1 - \frac{1}{2}\sqrt{1+4d}$ is the value of \hat{y} which solves $\ddot{x}(\frac{1}{2}, \hat{y}, -d) = 1$. That is, if $\hat{d} = -d$ and $\hat{y} \in [1 - \frac{1}{2}\sqrt{1+4d}, \frac{1}{2})$ then B is elected with probability one and if $\hat{d} = -d$ and $\hat{y} < 1 - \frac{1}{2}\sqrt{1+4d}$ then B is elected with a probability strictly smaller than one.

Similarly, $\sqrt{\frac{1}{4} - d}$ is the value of \hat{y} which solves $\ddot{x}(\frac{1}{4}, \hat{y}, d) = 0$. That is, if $\hat{d} = d$ and $\hat{y} \in [\sqrt{\frac{1}{4} - d}, \frac{1}{2})$ then B is elected with probability zero and if $\hat{d} = d$ and $\hat{y} < \sqrt{\frac{1}{4} - d}$ then B is elected with a probability strictly larger than zero.

Since $\sqrt{\frac{1}{4} - d}$ is strictly smaller than $1 - \frac{1}{2}\sqrt{1+4d}$ for any $d \in (0, \frac{1}{4})$, we must have that if $\hat{x} = 1/2$ then for all $\hat{y} \in [1 - \frac{1}{2}\sqrt{1+4d}, \frac{1}{2})$ candidate B 's payoff is equal to $1 - \hat{\rho}$.

Case 2. If B deviates to $\hat{y} \in [\sqrt{\frac{1}{4} - d}, 1 - \frac{1}{2}\sqrt{1+4d})$ then we have that if $\hat{d} = -d$ candidate B is elected with a probability strictly smaller than one and if $\hat{d} = d$ we have that candidate B is elected with probability zero. That is B 's payoff is strictly smaller than $1 - \hat{\rho}$.

Case 3. Finally, if B deviates to $\hat{y} < \sqrt{\frac{1}{4} - d}$ we have that $\ddot{x}(\frac{1}{2}, \hat{y}, -d) \in (0, 1)$ and $\ddot{x}(\frac{1}{2}, \hat{y}, d) \in (0, 1)$. We moreover observe that it is always the case that $\ddot{x}(\frac{1}{2}, \hat{y}, d) < 1 - \ddot{x}(\frac{1}{2}, \hat{y}, -d)$. This implies that $\hat{q}(\frac{1}{2}, \hat{y}, -d) < \hat{p}(\frac{1}{2}, \hat{y}, d) = 1 - \hat{q}(\frac{1}{2}, \hat{y}, d)$ and that $\hat{q}(\frac{1}{2}, \hat{y}, d) < \frac{1}{2}$.

We have to consider two cases:

(i) If $\hat{q}(\frac{1}{2}, \hat{y}, -d) \leq \frac{1}{2}$ with a straightforward Condorcet jury theorem like argument we obtain that $\lim_{n \rightarrow +\infty} \hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho}) < 1 - \hat{\rho}$.⁵ That is, there must exist \tilde{n} such that whenever $n > \tilde{n}$ deviations to $\hat{y} < \sqrt{\frac{1}{4} - d}$ are unprofitable for candidate B .

(ii) If $\hat{q}(\frac{1}{2}, \hat{y}, -d) > \frac{1}{2}$ we have that

$$\lim_{n \rightarrow +\infty} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k} = 0$$

and

$$\lim_{n \rightarrow +\infty} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k} = 1.$$

and, hence, $\lim_{n \rightarrow +\infty} \hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho}) = 1 - \hat{\rho}$.

⁵See, for example, Kirstein and Wangenheim, 2010.

What we need to show is that $\hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho})$ converges to $1 - \hat{\rho}$ from below because in that case there should exist \tilde{n} such that whenever $n > \tilde{n}$ deviations to $\hat{y} < \sqrt{\frac{1}{4} - d}$ would be unprofitable for candidate B .

To prove that $\hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho})$ converges to $1 - \hat{\rho}$ from below one needs to show that for n large enough we always have $\hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho}) < 1 - \hat{\rho}$. But $\hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho}) < 1 - \hat{\rho}$ holds if and only if

$$\hat{\rho} \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k} + (1 - \hat{\rho}) \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k} < 1 - \hat{\rho}$$

or if and only if

$$\frac{\hat{\rho}}{1 - \hat{\rho}} \frac{\sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k}}{1 - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k}} < 1.$$

We moreover notice that both $\sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k}$ and $1 - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k}$ converge to zero when n goes to infinity. By the Stolz–Cesaro theorem (see Stolz 1885 and Cesaro 1888) we have that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{\sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k}}{1 - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k}} = \\ & = \lim_{n \rightarrow +\infty} \frac{\sum_{k=\frac{n+2+1}{2}}^{n+2} \binom{n+2}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n+2-k} - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k}}{\sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k} - \sum_{k=\frac{n+2+1}{2}}^{n+2} \binom{n+2}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n+2-k}}. \end{aligned}$$

By Kirstein and Wagenheim (2010) we have that

$$\begin{aligned} & \sum_{k=\frac{n+2+1}{2}}^{n+2} \binom{n+2}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n+2-k} - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k} = \\ & = [2q(\frac{1}{2}, \hat{y}, d) - 1] \binom{n}{\frac{n+1}{2}} [\hat{q}(\frac{1}{2}, \hat{y}, d) (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))]^{\frac{n+1}{2}} < 0 \end{aligned}$$

and that

$$\begin{aligned} & \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k} - \sum_{k=\frac{n+2+1}{2}}^{n+2} \binom{n+2}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n+2-k} = \\ & = -[2q(\frac{1}{2}, \hat{y}, -d) - 1] \binom{n}{\frac{n+1}{2}} [\hat{q}(\frac{1}{2}, \hat{y}, -d) (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))]^{\frac{n+1}{2}} < 0. \end{aligned}$$

Thus,

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{\sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))^{n-k}}{1 - \sum_{k=\frac{n+1}{2}}^n \binom{n}{k} \hat{q}(\frac{1}{2}, \hat{y}, -d)^k (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))^{n-k}} = \\ & = \lim_{n \rightarrow +\infty} \frac{[2q(\frac{1}{2}, \hat{y}, d) - 1] \binom{n}{\frac{n+1}{2}} [\hat{q}(\frac{1}{2}, \hat{y}, d) (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))]^{\frac{n+1}{2}}}{-[2q(\frac{1}{2}, \hat{y}, -d) - 1] \binom{n}{\frac{n+1}{2}} [\hat{q}(\frac{1}{2}, \hat{y}, -d) (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))]^{\frac{n+1}{2}}} = \\ & = \frac{[2q(\frac{1}{2}, \hat{y}, d) - 1]}{-[2q(\frac{1}{2}, \hat{y}, -d) - 1]} \times \lim_{n \rightarrow +\infty} \left[\frac{\hat{q}(\frac{1}{2}, \hat{y}, d) (1 - \hat{q}(\frac{1}{2}, \hat{y}, d))}{\hat{q}(\frac{1}{2}, \hat{y}, -d) (1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))} \right]^{\frac{n+1}{2}} = 0 \end{aligned}$$

because as n increases $[\frac{\hat{q}(\frac{1}{2}, \hat{y}, d)(1 - \hat{q}(\frac{1}{2}, \hat{y}, d))}{\hat{q}(\frac{1}{2}, \hat{y}, -d)(1 - \hat{q}(\frac{1}{2}, \hat{y}, -d))}]^{\frac{n+1}{2}}$ converges to zero due to the fact that $\frac{1}{2} < \hat{q}(\frac{1}{2}, \hat{y}, -d) < 1 - \hat{q}(\frac{1}{2}, \hat{y}, d)$. That is, indeed $\hat{Q}_n(\frac{1}{2}, \hat{y}, \hat{\rho}) < 1 - \hat{\rho}$ for n large enough, therefore there must exist \tilde{n} such that whenever $n > \tilde{n}$ deviations to $\hat{y} < \sqrt{\frac{1}{4} - d}$ are unprofitable for candidate B .

Similarly, deviations of B to $\hat{y} > \hat{x} = \frac{1}{2}$ can be proven to be unprofitable by arguments symmetric to the ones presented here for the $\hat{y} < \hat{x} = \frac{1}{2}$ case. Finally, the arguments that rule out profitable deviations of A are identical to the ones presented here for B .

Notice that all the analysis that we have performed above would still hold, after appropriate modifications,⁶ if A located instead at $\frac{1}{2} \pm \varepsilon$ as long as $\varepsilon > 0$ were small enough. This is because a) all the arguments of cases 1 and 2 directly hold for $\hat{x} = \frac{1}{2} \pm \varepsilon$ when $\varepsilon > 0$ is small enough and b) all the inequalities of case 3 are strict and involve functions which are continuous in \hat{x} and \hat{y} and are, hence, preserved when \hat{x} takes values sufficiently close to $\frac{1}{2}$.

The second part of the proposition is a known result (see for example Aragonès and Xefteris, 2012).

■

⁶ Appropriate modifications here means to substitute threshold values. That is, to have $1 - \frac{1}{2}\sqrt{1 + 4d - 4\varepsilon + 4\varepsilon^2}$ instead of $1 - \frac{1}{2}\sqrt{1 + 4d}$ and $\sqrt{(\frac{1}{2} + \varepsilon)^2 - d}$ instead of $\sqrt{\frac{1}{4} - d}$.

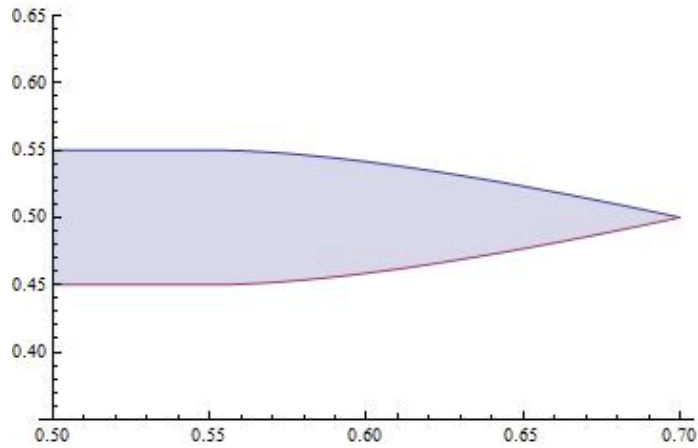


Figure 1: Minimizer strategies of A as a function of $\rho \in [1/2, 1/2+2d)$ when $d=0.1$.

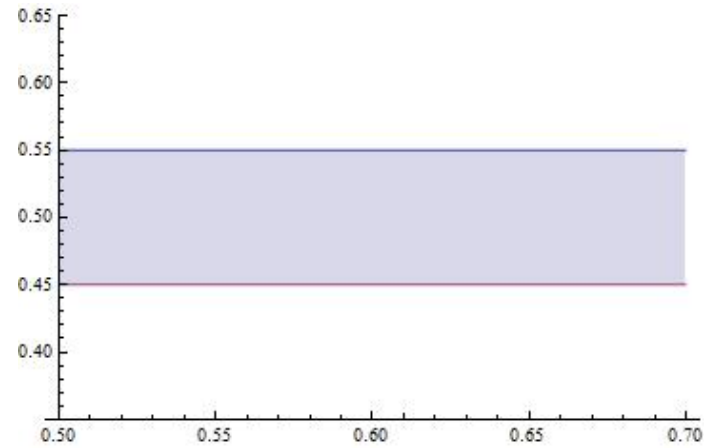


Figure 2: Minimizer strategies of B as a function of $\rho \in [1/2, 1/2+2d)$ when $d=0.1$.

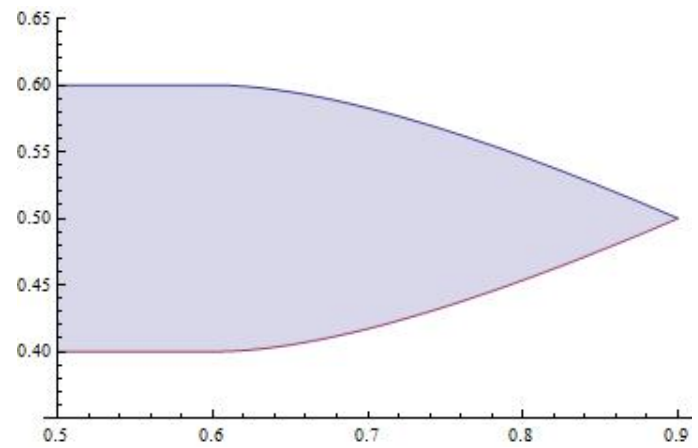


Figure 3: Minimizer strategies of A as a function of $\rho \in [1/2, 1/2+2d)$ when $d=0.2$.

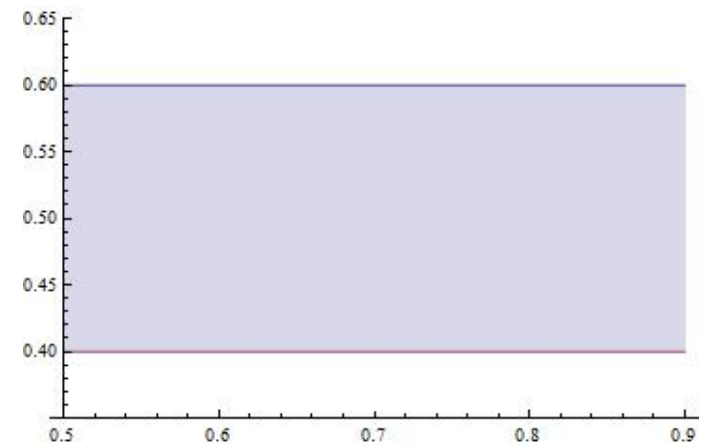


Figure 4: Minimizer strategies of B as a function of $\rho \in [1/2, 1/2+2d)$ when $d=0.2$.

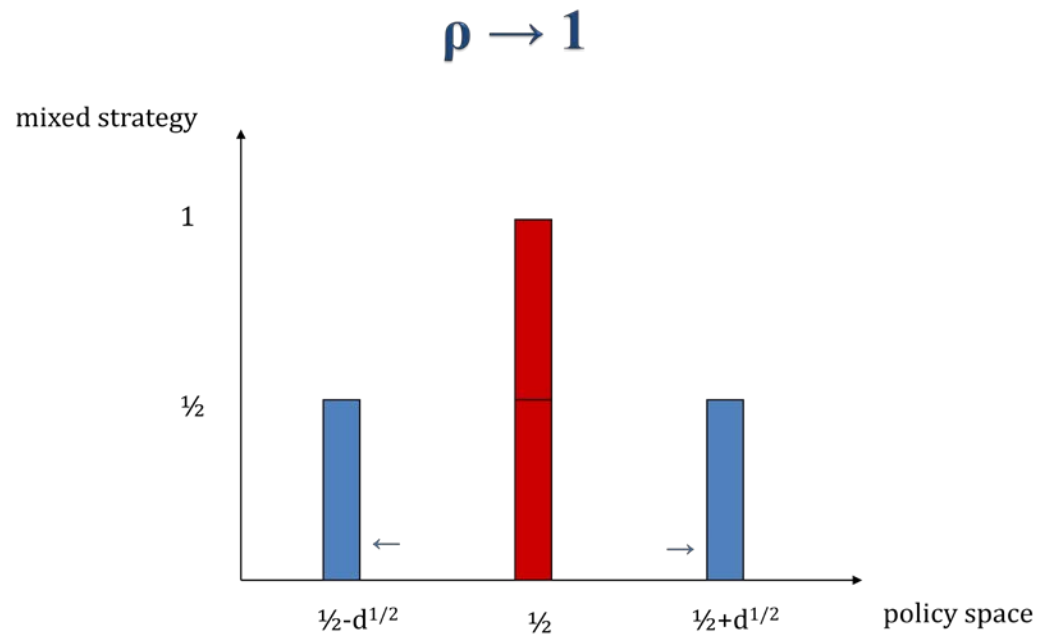


Figure 5: Mixed strategy equilibrium for $\rho = 1$.

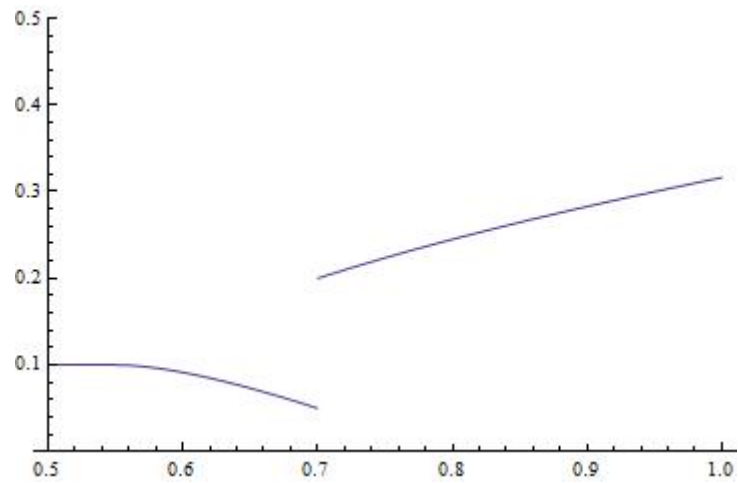


Figure 6: Maximum equilibrium differentiation as a function of ρ when $d=0.1$.

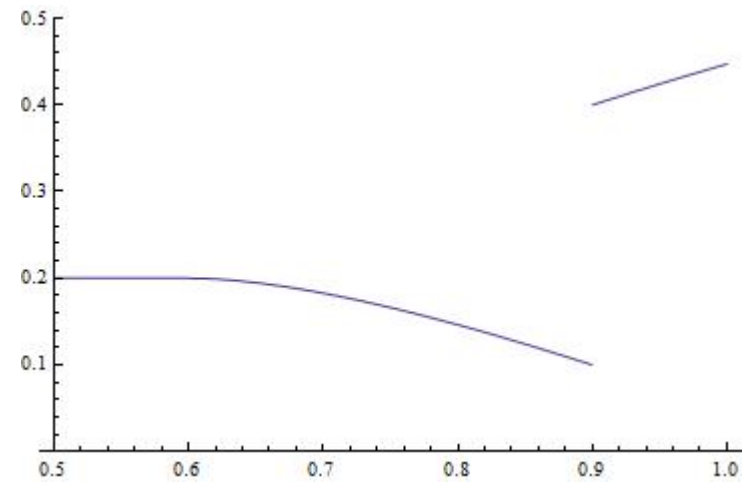


Figure 7: Maximum equilibrium differentiation as a function of ρ when $d=0.2$.