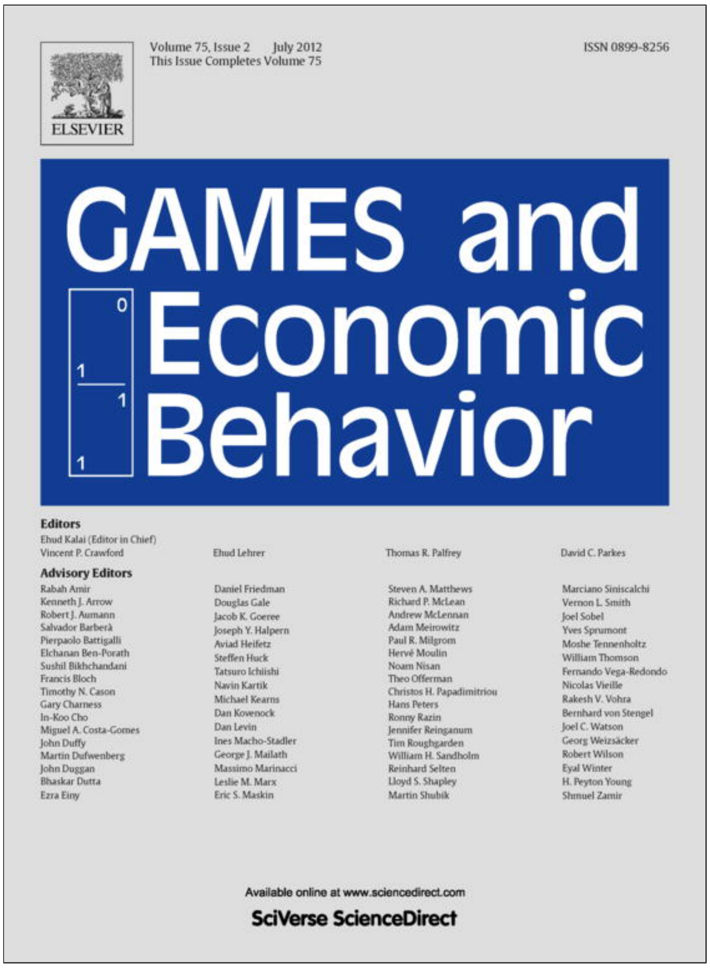

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Candidate quality in a Downsian model with a continuous policy space ☆

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ABSTRACT

This paper characterizes a unique mixed strategy Nash equilibrium in a one-dimensional Downsian model of two-candidate elections with a continuous policy space, where candidates are office motivated and one candidate enjoys a non-policy advantage over the other candidate. We show that if voters' utility functions are concave and the median voter ideal point is drawn from a unimodal distribution, there is a mixed strategy Nash equilibrium where the advantaged candidate chooses the ideal point of the expected median voter with probability one and the disadvantaged candidate uses a mixed strategy that is symmetric around it. Existence conditions require the variance of the distribution to be small enough relative to the size of the advantage.

1. Introduction

Candidate quality is considered to be a critical variable in electoral competition. It affects the decisions of politicians regarding whether to run for office, campaign fund-raising, voter behavior, election outcomes, and, ultimately, policy outcomes. Quality differences between two candidates can arise for many reasons, including charisma, office-holding experience, incumbency, advertising, and any other non-policy dimension that may affect the relative attractiveness of two candidates. In Political Science candidate quality is also denoted by "valence dimension" and its importance has been widely demonstrated over several decades of careful empirical research.[1]

All else constant, high quality candidates will fare better than low quality candidates. Furthermore, quality differences produce significant changes in the nature of political competition. The equilibrium properties of spatial competition between two candidates who differ in quality have been analyzed theoretically. Recent papers by Ansolabehere and Snyder (2000), Aragones and Palfrey (2002, 2004, and 2005), Groseclose (2001) and Hummel (2010) report a number of theoretical results about the equilibrium properties of spatial competition between two candidates who differ in quality.[2] The contribution of this paper is to complement the theoretical literature by extending the existing results.

The framework used in the theoretical literature to study the effect of candidate quality on political competition is based on the standard Downsian model with two candidates and an important twist: any voter will strictly prefer the "higher quality" candidate to the "lower quality" candidate if the candidates locate so that the voter is indifferent between the two candidates on the policy dimension.

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[1] See, e.g., Stokes (1963), Kiewiet (1983), and Kiewiet and Zheng (1993).

[2] There are also some earlier theoretical papers that studied related kinds of asymmetry, such as incumbency or partisanship, e.g., Adams (1999), Bernhardt and Ingberman (1985), and Londregan and Romer (1993).

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Groseclose (2001) shows that, for office motivated candidates, existence of pure strategy equilibrium is especially problematic for small-to-intermediate values of the quality advantage. The reason is that when the advantage is not very big the high quality candidate can win for sure by imitating the location of the low quality candidate, and the low quality candidate is always better off differentiating its location from the high quality one. These incentives do not allow for existence of pure strategy equilibria.

The theoretical research has been focused on the characterization of mixed strategy equilibrium in this set up. Aragones and Palfrey (2002) described the equilibrium strategies for a particular case: when the policy space is a finite grid of points on the $[0, 1]$ interval and when the advantage is relatively small. Hummel (2010) studies the same environment with the only difference that he considers that the advantage might be larger than in Aragones and Palfrey (2002). He characterizes the optimal actions of both candidates in a particular equilibrium of the game but, unlike Aragones and Palfrey (2002), he does not fully characterize any equilibrium of the game. The difficulty of the large advantage case is similar to the one of the continuous policy space, because in a continuous policy space any advantage size may be considered large, relative to the distance between two policies. In this paper we are able to deal partially with this problem on a continuous policy space by characterizing the unique equilibrium strategies for some parameter values.

The model we analyze follows the standard Downsian model with the following modifications: (1) we assume that voters have quadratic preferences instead of Euclidean and (2) we assume that the beliefs of candidates on the distribution of the median voter's ideal point are unimodal.

When the policy space is continuous and voters' preferences are Euclidean the payoff functions of the candidates are discontinuous (see, for example, Aragones and Palfrey, 2002). This discontinuity implies that the best response of the disadvantaged candidate is not well defined. By changing the voters' preferences from linear to quadratic the payoff functions of the candidates become continuous and the best response of the disadvantaged candidate becomes well defined. This assumption allows us to find conditions for existence of a mixed strategy equilibrium and also to describe the equilibrium strategies.

We describe a family of unimodal distributions for the median voter's ideal point that guarantees existence of an equilibrium in which the advantaged candidate chooses a pure strategy that concentrates all the probability in the expected location of the median voter, while the disadvantaged candidate chooses a mixed strategy that allocates equal probability to two policies that are symmetrically located around the expected median voter.

We find necessary and sufficient conditions for this equilibrium to exist. These conditions impose restrictions only on the size of the advantage relative to the variance of the candidates' beliefs on the voters' distribution of preferences. We find that, for any size of the advantage (small enough so that there is no pure strategy equilibrium) this equilibrium exists if and only if the level of uncertainty about the location of the median voter is low enough relative to the size of the advantage, that is, when candidates believe that the median voter's ideal point is close to 1/2 with high enough probability. We show that when this equilibrium exists it is unique. We also extend the analysis by replacing the assumption of quadratic to concave utility functions and we obtain the same qualitative results. Even though the existence of this equilibrium is restricted to a set of parameter values, we show that for any advantage size, no matter how small it is, we have a mixed strategy equilibrium of the described form if the variance of the distribution of the median voter is small enough. Otherwise, we show that in equilibrium both candidates use non-degenerate mixed strategies and the advantaged candidate chooses more moderate policies than the disadvantaged.

Notice that in this set up the advantaged candidate wants to imitate the policy choice of the disadvantaged so that voters use his quality advantage to differentiate between the two candidates and vote for him. However, what the advantaged candidate really needs is the vote of the median voter. If the median voter's ideal point is likely to be close to the policy choice of the disadvantaged candidate, then the best response for the advantaged candidate is easy: imitating his opponent he can defeat him with a policy that is also liked by the median voter. If the disadvantaged candidate is using a non-degenerated mixed strategy then this happens when the variance of the distribution of the median voter's ideal point is large enough.

Otherwise, when the median voter's ideal point is not likely to be close to the policy choice of the disadvantaged candidate, then the advantaged candidate faces a trade-off that is solved by the equilibrium strategies we propose. The uncertainty about the location of the median voter is the reason of the dilemma of the advantaged candidate. This tension is softened when the variance of the distribution of the median voter's ideal point becomes smaller, and it is in this case when we are able to completely characterize the equilibrium strategies.

As in similar models, we find that in equilibrium the advantaged candidate obtains a larger probability of winning than the disadvantaged candidate. We also find that as the value of the advantage becomes larger the probability of wining of the advantaged candidate increases, the equilibrium strategies of the two candidates are more differentiated, and the conditions of existence of equilibrium are relaxed, that is, the probability with which the median voter is around its expected position needed for an equilibrium to exist is smaller. Finally, as the value of the advantage becomes smaller, that is, as the difference between the two candidates vanishes, the optimal strategy of the disadvantaged candidate moves closer to the advantaged candidate's. That is, the model converges to the standard Downsian model and both players' equilibrium strategies converge to the expected median voter as candidate A's advantage shrinks to zero.

The rest of the paper proceeds as follows. The next section describes the formal model. Section 3 presents the derivation of the equilibrium strategies and analyzes its properties. Finally, Section 4 contains some concluding remarks.

2. The model

The policy space is the $[0, 1]$ interval. There are two candidates, A and D, who are referred to as the advantaged candidate and the disadvantaged candidate, respectively. Each candidate's objective is to maximize his probability of winning the election. There are n voters, an odd and finite number. We assume that candidates believe that the ideal point of each voter is an i.i.d. draw from a uniform distribution in $[0, 1]$.

Voters have a utility function with two components: a policy component, and a candidate image component. The policy component is characterized by an ideal point in the policy space, with utility of alternatives in the policy space a quadratic function of the distance between the ideal point and the location of the policy. The image component is captured by an additive constant ($d > 0$) to the utility a voter gets if the higher quality candidate wins the election.

The game takes place in two stages. In the first stage, candidates simultaneously choose positions in $[0, 1]$. In the second stage, voters vote for the candidate whose election would give them the highest utility. In case of indifference, a voter is assumed to vote for each candidate with probability equal to $1/2$.

Let x denote the policy position chosen by candidate A, and let y denote the policy position chosen by candidate D. Then, the utility that a voter with ideal point x_i obtains if A wins the election is given by $U_i(x) = d - (x_i - x)^2$ and his utility if candidate D wins is given by $U_i(y) = -(x_i - y)^2$, where $d > 0$ denotes the size of candidate A's advantage.

Since the behavior of the voters is unambiguous in this model, we define an equilibrium of the game only in terms of the location strategies of the two candidates in the first round. A pure strategy equilibrium is a pair of candidate locations (x, y) such that both candidates are maximizing the probability of winning, given the choices of the other candidate. A mixed strategy equilibrium is a pair of probability distributions (σ^A, σ^D) over $[0, 1]$ such that there is no mixed strategy for A that guarantees higher probability of winning than σ^A, given σ^D, and there is no mixed strategy for D that guarantees higher probability of winning than σ^D, given σ^A.

Notice that in this set up, if $x < y$ then all voters with $d - (x_i - x)^2 > -(x_i - y)^2$ prefer to vote for candidate A. Therefore, we have that all voters with an ideal point $x_i < \frac{x+y}{2} + \frac{d}{2(y-x)} = \hat{x}(x, y)$ prefer to vote for candidate A. Since the ideal point of each voter is drawn from a uniform distribution, this implies that the probability that a voter votes for the advantaged candidate is given by $p(x, y) = \min\{\hat{x}(x, y), 1\}$ and the probability that a voter votes for the disadvantaged candidate is given by $q(x, y) = \max\{0, 1 - \hat{x}(x, y)\}$.

Similarly if $x > y$ we have that all voters with an ideal point $x_i > \frac{x+y}{2} + \frac{d}{2(y-x)} = \hat{x}(x, y)$ prefer to vote for candidate A. This implies that the probability that a voter votes for the advantaged candidate is given by $p(x, y) = \min\{1, 1 - \hat{x}(x, y)\}$ and the probability that a voter votes for the disadvantaged candidate is given by $q(x, y) = \max\{0, \hat{x}(x, y)\}$.

Since voters' ideal points are random draws, the probability with which a candidate wins the election is given by the probability that he obtains the votes of at least a majority of the voters. Because each voter will vote for candidate A with probability $p(x, y)$ the probability with which candidate A is elected may be computed by the sum of the Bernoulli distributions corresponding to at least a majority of successes over n trials, that is,

$$P_n(x, y) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p(x, y)^k \big(1 - p(x, y)\big)^{n-k}.$$

Similarly we could also show that the probability with which D wins the election is given by

$$Q_n(x, y) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} q(x, y)^k \big(1 - q(x, y)\big)^{n-k} = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} \big(1 - p(x, y)\big)^k p(x, y)^{n-k} = 1 - P_n(x, y).$$

Observe that $p(x, y)$ and $q(x, y)$ are continuous functions of $x \in [0, 1]$ and $y \in [0, 1]$, and therefore $P_n(x, y)$ and $Q_n(x, y)$ are continuous functions of $x \in [0, 1]$ and $y \in [0, 1]$ as well. Finally, if $x = y$, we have that $p(x, y) = 1$ and $q(x, y) = 0$, that is, if both candidates choose the same location then the advantaged candidate wins with probability one, because in this case all voters would prefer to vote for him.

Since the voters' ideal points are represented by a sample of size n from a uniform distribution, then our observations can be ranked according to their corresponding value $z_1, z_2, \ldots, z_n$ such that $z_1 < z_2 < \cdots < z_n$. These z_i's are order statistics of a sample of size n drawn from a uniform distribution. Any such kth order statistic is itself a random variable distributed according to a Beta distribution with $a = k$ and $b = n + 1 - k$. This implies that the median of our sample, the $\frac{n+1}{2}$th order statistic, is distributed according to a Beta distribution with parameters $a = b = \frac{n+1}{2}$. The density function of such a Beta distribution is unimodal and symmetric around $\frac{1}{2}$ and its variance $Var(Beta(\frac{n+1}{2}, \frac{n+1}{2})) = \frac{1}{4n+8}$ decreases with n, that is, with a larger number of voters, the variance becomes smaller, and the probability that the ideal point of the median voter is close to $1/2$ increases. Therefore, this Beta distribution corresponds to the probability distribution of the median voter's ideal point, when we assume a finite and odd number of voters with ideal points being i.i.d. draws from a uniform probability distribution.[3]

[3] The payoff functions of the candidates in our set up also coincide with those of the Condorcet jury members (see, for example, Kirstein and Wagenheim, 2010). This coincidence will prove to be helpful for our analysis.

Notice that what we do is to describe the process that generates the beliefs of the candidates about the location of the median voter, instead of assuming directly a particular shape for the probability distribution that represents these beliefs. But we could have started by assuming that candidates had identical and common knowledge beliefs about the location of the median voter's ideal point represented by a symmetric and unimodal probability distribution. This assumption would lead to the exact same analysis and results that we have here. The description of the process that generates the distribution of the median voter's ideal point allows us to interpret the size of its variance as a function of the number of voters that participate in the election.

Finally, observe that if the location of the median voter's ideal point is uncertain from the point of view of the candidates, then the logical objective for candidates is to maximize the probability of winning. Alternatively one could also interpret our Beta distribution as a description of the ideal points of a continuum of voters. In this case, our analysis corresponds to candidates maximizing their vote share.

3. Equilibrium strategies

When $d = 0$, neither candidate has an advantage, and we are in the standard Downsian world, where in equilibrium the two candidates locate at $\frac{1}{2}$ and each wins with probability $\frac{1}{2}$. In general, when $d > 0$, there does not exist a pure strategy equilibrium. Different versions of this result have been stated and proven in Groseclose (2001) and Berger et al. (2000). The intuition is simple. If the disadvantaged candidate's location is perfectly predictable, the advantaged candidate can copy that strategy and win for sure. Therefore, at least the disadvantaged candidate must be mixing. This result is true unless d is sufficiently large so that candidate A can guarantee a payoff of 1 by choosing the location of the ideal point of the median voter.

Lemma 1. *If $d \geqslant \frac{1}{4}$ there is a pure strategy equilibrium in which A wins with probability one.*

(All proofs may be found in Appendix A.)

When the size of the advantaged is large enough, the utility that voters obtain from electing the advantage candidate is so large that cannot be compensated by the candidates' policy strategies. In this case, the advantaged candidate can guarantee a sure win if he chooses the expected median voter's ideal point, for any policy chosen by the disadvantaged candidate. Otherwise, if $d < \frac{1}{4}$ there is no pure strategy equilibrium. The aim of this paper is to show that if $d \in (0, \frac{1}{4})$ there exists a mixed strategy Nash equilibrium in which the advantaged candidate chooses a pure strategy and the disadvantaged candidate chooses a mixed strategy. In particular we show that in this equilibrium the advantaged candidate chooses a pure strategy corresponding to the ideal point of the expected median voter, $x = \frac{1}{2}$, while the disadvantaged candidate mixes between the two policy locations $y = \frac{1}{2} - \sqrt{d}$ and $y = \frac{1}{2} + \sqrt{d}$ each with equal probability. We find that this equilibrium exists as long as the number of voters is large enough relative to the size of the advantage d, that is, as long as the variance of the corresponding Beta distribution is small enough. We also find the minimal number of voters that guarantees existence of this Nash equilibrium as a function of the size of the advantage.

We start by demonstrating that the strategy proposed for candidate D, $\tilde{\sigma}^D = (y = \frac{1}{2} - \sqrt{d}$ with probability $\frac{1}{2}$ and $y = \frac{1}{2} + \sqrt{d}$ with probability $\frac{1}{2})$ is an optimal response to candidate A choosing $\tilde{\sigma}^A = \frac{1}{2}$. We prove that this holds true for all values of n.

Lemma 2. *For all $n > 0$ and for all $0 < d < \frac{1}{4}$ we have that $\tilde{\sigma}^D = (y = \frac{1}{2} - \sqrt{d}$ with probability 50% and $y = \frac{1}{2} + \sqrt{d}$ with probability 50%) is a best response to $\tilde{\sigma}^A = \frac{1}{2}$.*

The intuition of this result is straightforward: candidate D's payoff decreases as his policy moves away from the expected median voter's ideal point, and it also decreases whenever it moves close to A's policy choice. Since in this case both of them, the median voter's ideal point and the policy chosen by the advantaged candidate, coincide we have that candidate D wants to choose a policy that is far enough from 1/2 and close enough to 1/2. Because voters' utility is quadratic, candidates' payoff functions are continuous, and the best responses of candidate D are well defined. Thus, we obtain two interior solutions that are symmetric around 1/2.

Next we have to show that the strategy proposed for candidate A, $\tilde{\sigma}^A = \frac{1}{2}$, is a best response to candidate D choosing $\tilde{\sigma}^D = (y = \frac{1}{2} - \sqrt{d}$ with probability $\frac{1}{2}$ and $y = \frac{1}{2} + \sqrt{d}$ with probability $\frac{1}{2})$. Notice that when candidate D is choosing strategy $\tilde{\sigma}^D = (\frac{1}{2} - \sqrt{d}$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \sqrt{d}$ w.p. $\frac{1}{2})$ candidate A's probability of election is given by:

$$P_n\big(x, \tilde{\sigma}^D\big) = \frac{1}{2} P_n(x, 1/2 - \sqrt{d}) + \frac{1}{2} P_n(x, 1/2 + \sqrt{d})$$

where

$$P_n(x, 1/2 - \sqrt{d}) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p(x, 1/2 - \sqrt{d})^k \big(1 - p(x, 1/2 - \sqrt{d})\big)^{n-k}$$

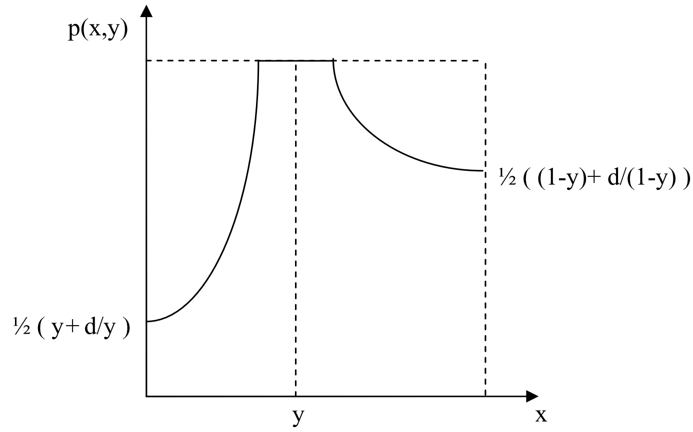

Fig. 1. The probability that a voter votes for the advantaged candidate, $p(x,y)$, as a function of the advantaged candidate's policy choice (x) given a policy choice of the disadvantaged candidate (y).

and

$$P_n(x, 1/2+\sqrt{d}) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p(x, 1/2+\sqrt{d})^k \bigl(1-p(x, 1/2+\sqrt{d})\bigr)^{n-k}.$$

Moreover, observe that the function $p(x,y)$ increases with x for all $x < 1-\sqrt{d-2y+y^2+1}$, remains constant ($p(x,y)=1$) for $x \in [1-\sqrt{d-2y+y^2+1}, \sqrt{y^2+d}]$ and decreases with x for all $x > \sqrt{y^2+d}$. (See Fig. 1.)

This implies that the probability that a voter prefers to vote for the advantaged candidate when the disadvantaged candidate chooses $1/2-\sqrt{d}$, that is $p(x, 1/2-\sqrt{d})$, is increasing in $x \in [0, 1-\theta^+)$, constant in $x \in [1-\theta^+, \theta^-]$ and decreasing in $x \in (\theta^-, 1]$, where $\theta^+ = \sqrt{\frac{1}{4}+2d+\sqrt{d}}$ and $\theta^- = \sqrt{\frac{1}{4}+2d-\sqrt{d}}$. Similarly $p(x, 1/2+\sqrt{d})$ is increasing in $x \in [0, 1-\theta^-)$, constant in $x \in [1-\theta^-, \theta^+]$ and decreasing in $x \in (\theta^+, 1]$. Since $\theta^- < 1/2$ for any $d \in (0, \frac{1}{4})$ and $\frac{\partial P_n(x,y)}{\partial p} > 0$, it naturally follows that $P_n(x, \tilde{\sigma}^D) = \frac{1}{2}P_n(x, 1/2-\sqrt{d}) + \frac{1}{2}P_n(x, 1/2+\sqrt{d})$ is increasing in $x \in [0, \theta^-)$ and decreasing in $x \in (1-\theta^-, 1]$ for any $n > 0$. This implies that when D chooses the proposed mixed strategy $\tilde{\sigma}^D$ the best response of A must belong to $[\theta^-, 1-\theta^-]$ for any $n > 0$. Thus we only have to investigate the shape of A's payoff function in the interval around 1/2 corresponding to $[\theta^-, 1-\theta^-]$.

We first analyze the case of a single voter, that is, $n = 1$, which corresponds to an expected median voter's ideal point drawn from a uniform distribution on $[0, 1]$. In this case, we find that the proposed strategies do not form an equilibrium. However this result will help us to illustrate the path that we will follow to demonstrate the main result. Since we know that the proposed strategy for candidate D is a best response for all values of n, then it must be the case that the proposed strategy for candidate A is not.

Lemma 3. *If $n = 1$ and $0 < d < \frac{1}{4}$ then $x = \frac{1}{2}$ is not a best response for A to $\tilde{\sigma}^D$.*

The specific shape of the payoff function of the advantaged candidate for the case $n = 1$ illustrates the reason why $x = \frac{1}{2}$ is not a best response (see Fig. 2). For $n = 1$ the payoff function is convex in $[\theta^-, 1-\theta^-]$ and thus the function $P_n(x, \tilde{\sigma}^D)$ achieves a local minimum at $x = \frac{1}{2}$. This will also be the case for small enough values of n. Recall that small values of n imply a large variance of the probability distribution of the median voter's ideal point. This is the case in which the median voter's ideal point is likely to be close to the policies chosen by the disadvantaged candidate, and thus the best responses of the advantage candidate will also be close to them.

But as n increases the values of the payoff function in the interval $[\theta^-, 1-\theta^-]$ increase and for n large enough the function becomes concave exhibiting a global maximum at $\frac{1}{2}$. (See Figs. 3, 4 and 5.) Increasing the number of voters reduces the variance of the probability distribution of the median voter's ideal point, and it becomes less likely to find the median voter's ideal point close to the policies chosen by the disadvantaged candidate, and more likely for it to be around 1/2. As the variance becomes smaller, the best responses of the advantage candidate concentrate at 1/2.

The more likely it is that the median voter is close to 1/2 the more profitable this policy is for the advantaged candidate. As the payoff he obtains from policies in $[\theta^-, 1-\theta^-]$ increases, his payoff function becomes concave in this interval and for n large enough it shows a unique maximum. In order to prove that $\tilde{\sigma}^A = 1/2$ is a global maximum of $P_n(x, \tilde{\sigma}^D)$ we have to show that $P_n(x, \tilde{\sigma}^D)$ is increasing for all $x \in [\theta^-, 1/2]$ and decreasing for all $x \in [1/2, 1-\theta^-]$. We will show that as n increases the slope of $P_n(x, \tilde{\sigma}^D)$ will gradually become positive for $x \in [\theta^-, 1/2]$ and negative for $x \in [1/2, 1-\theta^-]$, while the sign of the slope everywhere else does not change, which will lead to the conclusion that $\tilde{\sigma}^A = 1/2$ is indeed the unique best response of A to $\tilde{\sigma}^D$ for some large n.

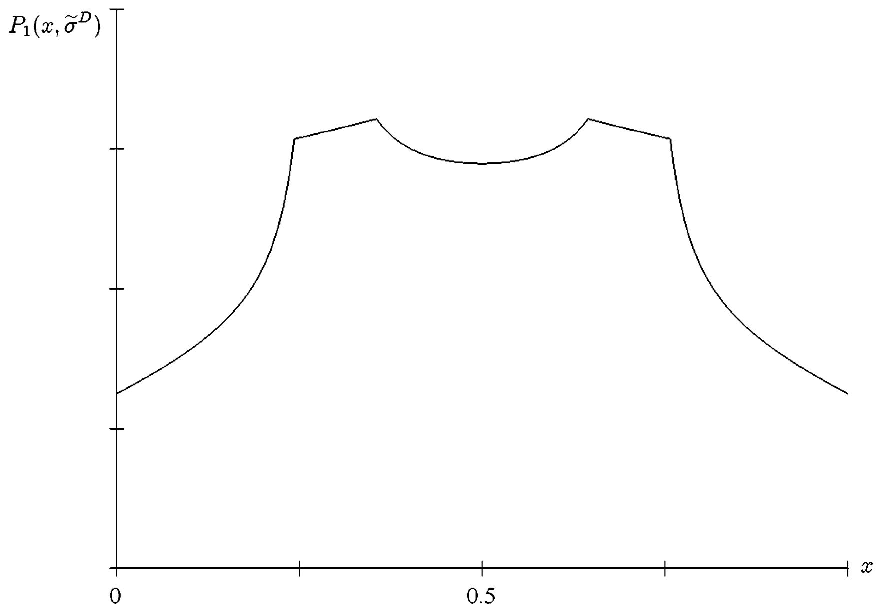

Fig. 2. The probability that candidate A wins, $P_1(x, \sigma^D)$, as a function of the advantaged candidate's policy choice (x) given the best response of the disadvantaged candidate (σ^D) when $n = 1$ and $d = 0.05$.

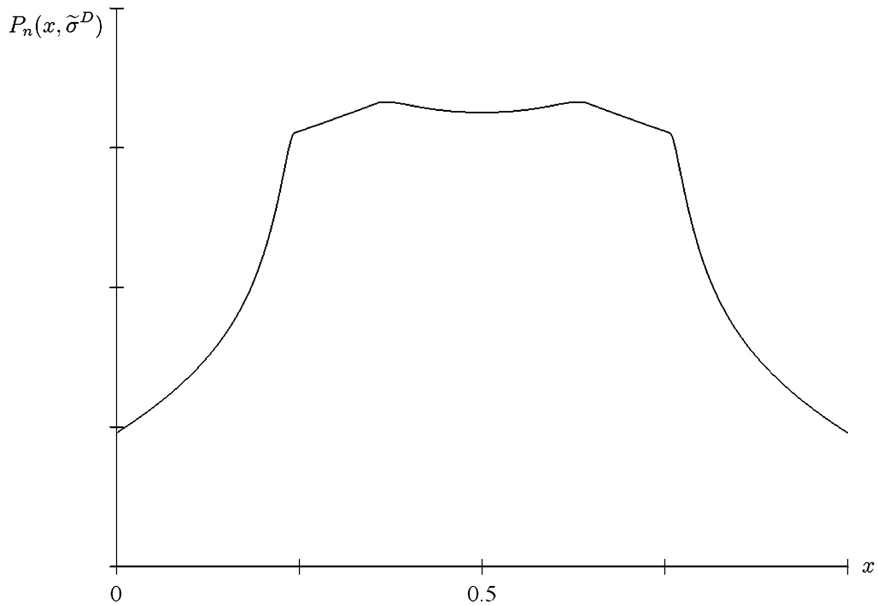

Fig. 3. The probability that candidate A wins, $P_n(x, \sigma^D)$, as a function of the advantaged candidate's policy choice (x) given the best response of the disadvantaged candidate (σ^D) when $n = 3$ and $d = 0.05$ thus $n < 1/4d$.

Lemma 4. *For n large enough we have that* $\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x} > 0$ *for* $x \in [\theta^-, \frac{1}{2})$ *and* $\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x} < 0$ *for* $x \in (\frac{1}{2}, 1 - \theta^-]$.

This concludes the proof that $x = 1/2$ is the best response of the advantaged candidate when the disadvantaged plays $\tilde{\sigma}^D$ if n is large enough. Combining the results described above we have proven that a mixed strategy equilibrium exists for large enough values of n. Analyzing the changes of the shape of candidate A's payoff function in the interval $[\theta^-, 1 - \theta^-]$ we can find the exact value of n, as a function of d, for which this function becomes flat. This value is what determines the necessary and sufficient conditions for existence of the equilibrium, because the payoff function becomes concave above it and convex below it.

Lemma 5. $x = \frac{1}{2}$ *is a global maximum of* $P_n(x, \tilde{\sigma}^D)$ *if and only if* $n \geqslant \frac{1}{4d}$.

Thus we have characterized the necessary and sufficient conditions for $\tilde{\sigma}^A = \frac{1}{2}$ and $\tilde{\sigma}^D = (y = \frac{1}{2} - \sqrt{d}$ with probability $\frac{1}{2}$ and $y = \frac{1}{2} + \sqrt{d}$ with probability $\frac{1}{2})$ to be equilibrium strategies. Given all the results developed above we can now state the main result in the following proposition.

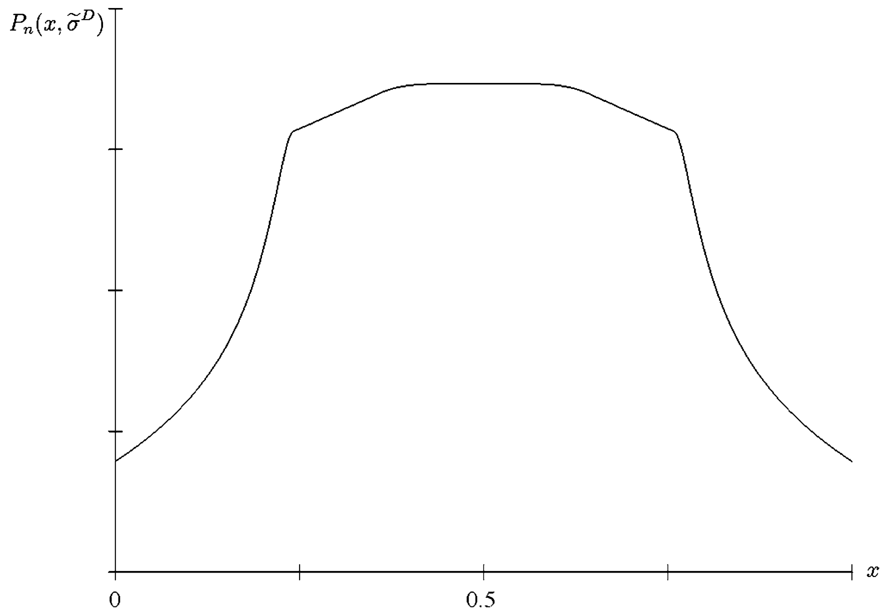

Fig. 4. The probability that candidate A wins, $P_n(x, \sigma^D)$, as a function of the advantaged candidate's policy choice (x) given the best response of the disadvantaged candidate (σ^D) when $n = 5$ and $d = 0.05$ thus $n = 1/4d$.

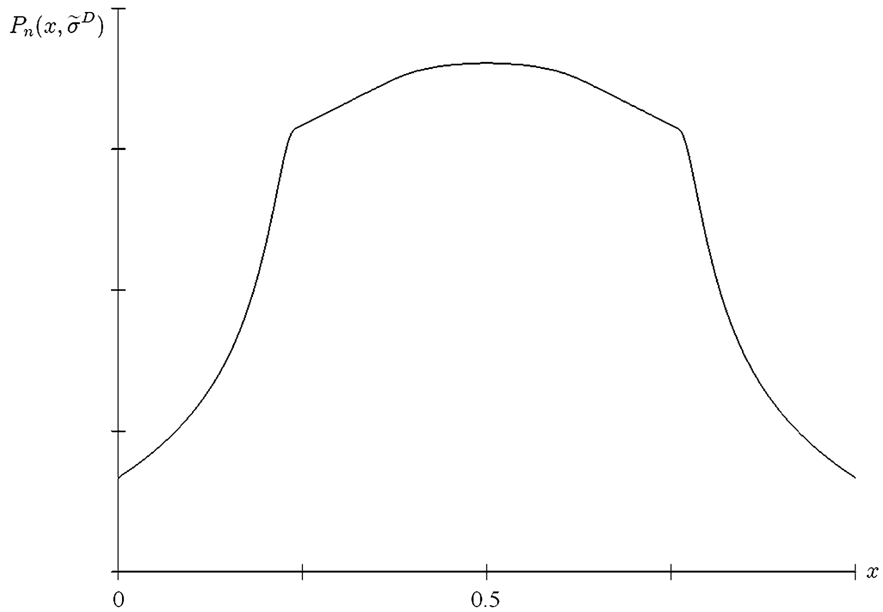

Fig. 5. The probability that candidate A wins, $P_n(x, \sigma^D)$, as a function of the advantaged candidate's policy choice (x) given the best response of the disadvantaged candidate (σ^D) when $n = 7$ and $d = 0.05$ thus $n > 1/4d$.

Proposition 1. *The profile of mixed strategies $\tilde{\sigma}^A = \frac{1}{2}$ and $\tilde{\sigma}^D = (\frac{1}{2} - \sqrt{d}$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \sqrt{d}$ w.p. $\frac{1}{2})$ constitutes a Nash equilibrium if and only if $n \geqslant \frac{1}{4d}$.*

The size of the non-policy advantage, d, determines the minimal size of the electorate that is needed in order to guarantee the existence of the proposed equilibrium. The smaller the advantage, the larger the values of n for which this equilibrium exists. In terms of our interpretation with the Beta distribution, for smaller values of the advantage we need the distribution of the ideal point of the median voter to be more concentrated around 1/2 for this equilibrium to exist, that is we need the variance of the Beta distribution to be smaller. Since $Var = \frac{1}{4n+8}$ in order to have $n \geqslant \frac{1}{4d}$ we need that $Var \leqslant \frac{d}{1+8d}$.

Notice that for smaller values of d the optimal strategy of the disadvantaged candidate moves closer to the advantaged candidate's optimal strategy, and both players' equilibrium strategies converge to the expected median voter's ideal point as candidate A's advantage shrinks to zero. On the contrary, as the advantage increases, candidates' differentiation increases, the equilibrium strategies of the disadvantaged candidate move away from 1/2 and the restriction on the variance is relaxed.

We also have that in equilibrium the probability with which the advantaged candidate wins is always larger than 1/2 and therefore larger than the disadvantaged's probability of winning. And as the value of the advantage increases the probability of winning of the advantaged candidate also increases. Observe that since by symmetry we have that $P_n(\frac{1}{2}, 1/2 - \sqrt{d}) = P_n(\frac{1}{2}, 1/2 + \sqrt{d})$, then we also have that $P_n(\frac{1}{2}, \tilde{\sigma}^D) = P_n(\frac{1}{2}, 1/2 - \sqrt{d})$. And since $P_n(\frac{1}{2}, 1/2 - \sqrt{d})$ is increasing in $p(\frac{1}{2}, 1/2 - \sqrt{d})$ and $p(\frac{1}{2}, 1/2 - \sqrt{d})$ is increasing in d, it follows that $P_n(\frac{1}{2}, \tilde{\sigma}^D)$ is increasing in d as well. Thus, for any value of n, the equilibrium payoff of the advantaged candidate increases with d.

In equilibrium we also have that increasing the number of voters for a fixed value of the advantage, that is decreasing uncertainty, leads to higher payoffs for the advantaged candidate,[4] as in Aragones and Palfrey (2004). And when the number of voters tends to infinity, uncertainty disappears, and we have a pure strategy equilibrium with the advantaged candidate locating at the ideal point of the median voter winning for sure against any strategy of the disadvantaged candidate. We also find that the level of uncertainty does not affect the equilibrium strategies, but increasing uncertainty may lead to non-existence of equilibrium.

Finally, we show that whenever this equilibrium exists it is unique. Since we are analyzing a two player constant sum game we know that the equilibrium payoffs are unique. The continuity of the payoff functions, obtained from the quadratic utility functions, allows for well defined best responses, and the uniqueness result follows.

Proposition 2. *The profile of mixed strategies $\tilde{\sigma}^A = \frac{1}{2}$ and $\tilde{\sigma}^D = (\frac{1}{2} - \sqrt{d}$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \sqrt{d}$ w.p. $\frac{1}{2})$ is the unique Nash equilibrium of the game when $n \geqslant \frac{1}{4d}$.*

The existence of this equilibrium is only guaranteed for a set of parameter values. However it is guaranteed for any size of the advantage that is small enough so that no pure strategy equilibrium exists. We have proven that given any value of the advantage, there exists a class of distributions of the median voter's ideal point for which this equilibrium exists. It is also true that the size of this class of distributions decreases when d decreases but for any positive value of d it is non-vanishing.

Since in our case existence of an equilibrium and uniqueness of the equilibrium payoff for both players is guaranteed for any value of the parameters, if $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D\}$ represents the strategies of an equilibrium when $n < \frac{1}{4d}$, and $[s^A, S^A] \subseteq [0,1]$ and $[s^D, S^D] \subseteq [0,1]$ represent the support of $\boldsymbol{\sigma}^A$ and $\boldsymbol{\sigma}^D$ respectively, then we can state the following.

Proposition 3. *If $n < \frac{1}{4d}$ and $d < 1/4$, in any equilibrium we have that*

(a) *$s^A < S^A$ and $s^D < S^D$ (that is, both players mix),*
(b) *$[s^A, S^A] \subset [s^D, S^D]$ (that is, the support of candidate A's strategy is strictly smaller than the support of candidate D) and*
(c) *$P_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) \in [P_n(\tilde{\sigma}^A, \tilde{\sigma}^D), \frac{1}{2} + \frac{1}{2} P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)]$ and $Q_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) \in [\frac{1}{2} Q_n(\tilde{\sigma}^A, \tilde{\sigma}^D), Q_n(\tilde{\sigma}^A, \tilde{\sigma}^D)]$.*

This proposition shows that for $n < \frac{1}{4d}$ in equilibrium both players are using non-degenerate mixed strategies. Observe that if A is using a pure strategy it has to be $x \neq 1/2$ and in that case D has a unique best response ($y = x - \sqrt{d}$ if $x > \frac{1}{2}$ and $y = x + \sqrt{d}$ if $x < \frac{1}{2}$). Since we know that a pure strategy equilibrium does not exist, in equilibrium we cannot have $x \neq \frac{1}{2}$. Thus, it has to be the case that A must be using a mixed strategy as well.

This proposition also offers some information about the bounds of the supports of the equilibrium strategies: the support of the advantaged candidate has to be more moderate than that of the disadvantaged candidate. Following the same lines of intuition that we obtained for the equilibrium described before, in this case we find that the large variance of the distribution of the median voter's ideal point drives the advantaged candidate to choose policies that are close to the ones chosen by the disadvantaged candidate, but not beyond them.

Concave utility functions

We now extend some of the previous results to a more general family of utility functions. If we consider strictly concave utility functions we can still prove that for a range of parameter values a mixed strategy equilibrium of the described form exists. In this case the equilibrium strategies depend on the concavity of the utility function, but the main features follow the lines of the equilibrium strategies found for the quadratic case.

Suppose that the utility that a voter with ideal point x_i obtains if A wins the election is given by $U_i(x) = d - \phi(|x_i - x|)$ and his utility if candidate D wins is given by $U_i(y) = -\phi(|x_i - y|)$. We assume that the function $\phi(.)$ is twice differentiable, strictly increasing for any positive value, strictly convex everywhere and that $\phi(0) = 0$ and $\phi'(0) = 0$.

Proposition 4.

(a) *If $d \geqslant \phi(\frac{1}{2})$ there is a pure strategy equilibrium in which A wins with probability one.*
(b) *If $d < \phi(\frac{1}{2})$ there is no pure strategy equilibrium.*

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4 Notice that from the Condorcet Jury Theorem we have that $P_n(\frac{1}{2}, 1/2 - \sqrt{d})$ is increasing in n whenever $p(\frac{1}{2}, 1/2 - \sqrt{d}) > 1/2$.

(c) *When $d < \phi(\frac{1}{2})$ the profile of mixed strategies $\tilde{\sigma}^A = \frac{1}{2}$ and $\tilde{\sigma}^D = (\frac{1}{2} - \phi^{-1}(d)$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \phi^{-1}(d)$ w.p. $\frac{1}{2})$ is a Nash equilibrium of the game for n sufficiently large.*

Thus we have that for any strictly concave utility function the same type of equilibrium exists in which the advantaged candidate chooses a pure strategy corresponding to the expected median voter's ideal point, and the disadvantaged candidate mixes between two policies that are symmetrically located around it. This means that the results found for quadratic utility functions do not represent a knife edge case. Hummel (2010) considers a discrete policy space and linear utilities and proves that, for any symmetric and unimodal distribution of the median voter's ideal point, when the number of policies tends to infinity the mixed strategy of both candidates have a non-vanishing support. We know that this is not the case for concave utility functions, thus the linear case appears to be the knife edge case.

4. Concluding remarks

The main contribution of this paper is to characterize the unique equilibrium strategies for a Downsian model with an advantaged candidate when the policy space is continuous. We have shown that the features of this equilibrium are in line with those of the equilibria found for similar models.

As in Aragones and Palfrey (2002) and Hummel (2010) in our equilibrium the advantaged candidate selects more centrist strategies than the disadvantaged candidate, and also the equilibrium strategies change with changes in the value of the advantage: they differentiate more whenever the advantage is larger. We also replicate their other equilibrium results: the advantaged candidate always wins with larger probability, and this probability increases with the value of the advantage and decreases with the level of uncertainty.

The contribution of this paper is to extend the results that were known to broader set-up that include continuous policy space and concave utility functions. The consideration of a continuous policy space is basic for the literature based on Downsian models and it was nonexistent for the case of an advantaged candidate. The results we produce fill up the existing gap in it. The robustness of the analysis of concave utility functions revealed the weaknesses of the use of linear utility functions in similar models.

Appendix A

Proof of Lemma 1. Notice that if $d \geqslant \frac{1}{4}$ and the advantaged selects a pure strategy $x = \frac{1}{2}$ then a voter with ideal point x_i prefers to vote for the advantaged candidate if and only if $d - (x_i - \frac{1}{2})^2 > -(x_i - y)^2$ which is equivalent to $x_i(1 - 2y) > -y^2 - d + \frac{1}{4}$. For any strategy of the disadvantaged candidate $y < \frac{1}{2}$ we have that $x_i(1 - 2y) > 0 > -y^2 - d + \frac{1}{4}$ for all x_i, thus all voters prefer to vote for candidate A. Similarly, for any strategy of the disadvantaged candidate $y > \frac{1}{2}$ we have that all voters with $x_i < \frac{y^2+d-\frac{1}{4}}{2y-1}$ prefer to vote for candidate A, but since $d \geqslant \frac{1}{4}$ implies $\frac{y^2+d-\frac{1}{4}}{2y-1} > \frac{y^2}{2y-1} > 1$ for all $y \in [0, 1]$, we have that all voters prefer to vote for candidate A. Thus, if $d \geqslant \frac{1}{4}$ there is a pure strategy equilibrium in which the advantaged candidate obtains the votes from all voters. Therefore, A wins with probability one. □

Proof of Lemma 2. Given that $x = \frac{1}{2}$, we search for a value of y that maximizes the payoff function of the disadvantaged candidate, that is, the following expression:

$$Q_n\left(\frac{1}{2}, y\right) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} q\left(\frac{1}{2}, y\right)^k \left(1 - q\left(\frac{1}{2}, y\right)\right)^{n-k}.$$

Kirstein and Wagenheim (2010) show that $\frac{\partial Q_n(x,y)}{\partial q} = n\binom{n-1}{\frac{n-1}{2}}[q(x, y)(1 - q(x, y))]^{\frac{n-1}{2}} > 0$ for $x \geqslant \frac{1}{2}$. Thus we have that $Q_n(\frac{1}{2}, y)$ is strictly increasing in $q(\frac{1}{2}, y)$.

Therefore, in order to find a value of y that maximizes $Q_n(\frac{1}{2}, y)$ it is enough to find the values $y < 1/2$ that maximize $q(\frac{1}{2}, y) = \max\{0, \hat{x}(\frac{1}{2}, y)\} = \max\{0, \frac{1}{4} + \frac{y}{2} - \frac{d}{1-2y}\}$; and the values $y > 1/2$ that maximize $q(\frac{1}{2}, y) = \max\{0, 1 - \hat{x}(x, y)\} = \max\{0, \frac{3}{4} - \frac{y}{2} + \frac{d}{1-2y}\}$.

Notice that $1/2 - \sqrt{d} = \arg\max \frac{1}{4} + \frac{y}{2} - \frac{d}{1-2y}$, and $1/2 - \sqrt{d} \in [0, \frac{1}{2}]$ when $d \in (0, \frac{1}{4}]$. Similarly, when $y > 1/2$ we find that $1/2 + \sqrt{d} = \arg\max \frac{3}{4} - \frac{y}{2} + \frac{d}{1-2y}$ and $1/2 + \sqrt{d} \in [\frac{1}{2}, 1]$ when $d \in (0, \frac{1}{4}]$.

Therefore, the proposed mixed strategy for candidate D is a best response to $x = 1/2$ for any $n > 0$. □

Proof of Lemma 3. If $n = 1$ candidate A's probability of winning is given by

$$P_1\left(x, \tilde{\sigma}^D\right) = \frac{1}{2} p(x, 1/2 - \sqrt{d}) + \frac{1}{2} p(x, 1/2 + \sqrt{d}).$$

We have seen that $P_1(x, \tilde{\sigma}^D)$ is increasing in $x \in [0, \sqrt{\frac{1}{4} + 2d - \sqrt{d}})$ and that $P_1(x, \tilde{\sigma}^D)$ is decreasing in $x \in (1 - \sqrt{\frac{1}{4} + 2d - \sqrt{d}}, 1]$ and, therefore, the best response of A must belong to $[\sqrt{1/4 + 2d - \sqrt{d}}, 1 - \sqrt{1/4 + 2d - \sqrt{d}}]$.

Observe that for all $d \in (0, \frac{1}{4})$ we have that $1/2 - \sqrt{d} < \sqrt{1/4 + 2d - \sqrt{d}} < 1/2 < 1 - \sqrt{1/4 + 2d - \sqrt{d}} < 1/2 + \sqrt{d}$ and that if $n = 1$ candidate A's probability of election $P_1(x, \tilde{\sigma}^D) = \frac{1}{2} p(x, 1/2 - \sqrt{d}) + \frac{1}{2} p(x, 1/2 + \sqrt{d})$ can also be written as

$$P_1(x, \tilde{\sigma}^D) = 1/2\left(1 - \frac{x + 1/2 - \sqrt{d}}{2} + \frac{d}{2(x - 1/2 + \sqrt{d})}\right) + 1/2\left(\frac{x + 1/2 + \sqrt{d}}{2} - \frac{d}{2(x - 1/2 - \sqrt{d})}\right)$$
$$\text{for } x \in \left[\sqrt{1/4 + 2d - \sqrt{d}}, 1 - \sqrt{1/4 + 2d - \sqrt{d}}\right].$$

Thus, $\frac{\partial P_1(x, \tilde{\sigma}^D)}{\partial x} = -\frac{d}{4(x - 1/2 + \sqrt{d})^2} + \frac{d}{4(x - 1/2 - \sqrt{d})^2} < 0$ if and only if $(x - 1/2 + \sqrt{d})^2 < (x - 1/2 - \sqrt{d})^2$. This implies that $P_1(x, \tilde{\sigma}^D)$ is decreasing for $x \in [\sqrt{1/4 + 2d - \sqrt{d}}, 1/2)$ and it is increasing for $x \in (1/2, 1 - \sqrt{1/4 + 2d - \sqrt{d}}]$. See Fig. 2.

Therefore $P_1(x, \tilde{\sigma}^D)$ is increasing for $x \in [0, \sqrt{1/4 + 2d - \sqrt{d}}]$, decreasing for $x \in [\sqrt{1/4 + 2d - \sqrt{d}}, 1/2)$, increasing for $x \in (1/2, 1 - \sqrt{1/4 + 2d - \sqrt{d}}]$ and decreasing for $x \in [1 - \sqrt{1/4 + 2d - \sqrt{d}}, 1]$. This implies that when $n = 1$ the optimal responses of candidate A are either $x = \sqrt{1/4 + 2d - \sqrt{d}}$ or $x = 1 - \sqrt{1/4 + 2d - \sqrt{d}}$ but not $x = \frac{1}{2}$. □

Proof of Lemma 4. Let's show that $\frac{\partial P_n(x, \tilde{\sigma}^D)}{x} > 0$ for $x \in [\sqrt{1/4 + 2d - \sqrt{d}}, \frac{1}{2})$. A similar analysis would prove that $\frac{\partial P_n(x, \tilde{\sigma}^D)}{x} < 0$ for $x \in (\frac{1}{2}, 1 - \sqrt{1/4 + 2d - \sqrt{d}}]$.

Since $P_n(x, \tilde{\sigma}^D) = \frac{1}{2} P_n(x, 1/2 - \sqrt{d}) + \frac{1}{2} P_n(x, 1/2 + \sqrt{d})$ we have that

$$\frac{\partial P_n(x, \tilde{\sigma}^D)}{\partial x} = \frac{1}{2} \frac{\partial P_n(x, 1/2 - \sqrt{d})}{\partial x} + \frac{1}{2} \frac{\partial P_n(x, 1/2 + \sqrt{d})}{\partial x}$$

where

$$P_n(x, 1/2 - \sqrt{d}) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p(x, 1/2 - \sqrt{d})^k \left(1 - p(x, 1/2 - \sqrt{d})\right)^{n-k}$$

and

$$P_n(x, 1/2 + \sqrt{d}) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p(x, 1/2 + \sqrt{d})^k \left(1 - p(x, 1/2 + \sqrt{d})\right)^{n-k}.$$

We can compute the derivative of $P_n(x, 1/2 - \sqrt{d})$ with respect to $p(.)$ using the results in Kirstein and Wagenheim (2010) and obtain $\frac{\partial P_n(x, 1/2 - \sqrt{d})}{\partial p} = n\binom{n-1}{\frac{n-1}{2}}[p(x, 1/2 - \sqrt{d})(1 - p(x, 1/2 - \sqrt{d}))]^{\frac{n-1}{2}}$. Thus, we have that the total derivative of $P_n(x, 1/2 - \sqrt{d})$ with respect to x is $\frac{\partial P_n(x, 1/2 - \sqrt{d})}{\partial x} = \frac{\partial P_n(x, 1/2 - \sqrt{d})}{\partial p} \frac{\partial p(x, 1/2 - \sqrt{d})}{\partial x}$ which can be written as

$$\frac{\partial P_n(x, 1/2 - \sqrt{d})}{\partial x} = n\binom{n-1}{\frac{n-1}{2}}\left[p(x, 1/2 - \sqrt{d})\left(1 - p(x, 1/2 - \sqrt{d})\right)\right]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 - \sqrt{d})}{\partial x}$$

and similarly we have that

$$\frac{\partial P_n(x, 1/2 + \sqrt{d})}{\partial x} = n\binom{n-1}{\frac{n-1}{2}}\left[p(x, 1/2 + \sqrt{d})\left(1 - p(x, 1/2 + \sqrt{d})\right)\right]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 + \sqrt{d})}{\partial x}.$$

Therefore,

$$\frac{\partial P_n(x, \tilde{\sigma}^D)}{\partial x} = \frac{n}{2}\binom{n-1}{\frac{n-1}{2}}\Bigg[\left[p(x, 1/2 - \sqrt{d})\left(1 - p(x, 1/2 - \sqrt{d})\right)\right]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 - \sqrt{d})}{\partial x}$$
$$+ \left[p(x, 1/2 + \sqrt{d})\left(1 - p(x, 1/2 + \sqrt{d})\right)\right]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 + \sqrt{d})}{\partial x}\Bigg]$$

and we need to prove that for large enough values of n we have that

$$\left[p(x,1/2-\sqrt{d})\left(1-p(x,1/2-\sqrt{d})\right)\right]^{\frac{n-1}{2}}\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}$$
$$+\left[p(x,1/2+\sqrt{d})\left(1-p(x,1/2+\sqrt{d})\right)\right]^{\frac{n-1}{2}}\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}>0$$

whenever $x\in[\sqrt{1/4+2d-\sqrt{d}},\frac{1}{2})$.

This holds if and only if

$$\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]^{\frac{n-1}{2}}\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}+\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}>0.$$

First of all we will show that $[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}]^{\frac{n-1}{2}}$ decreases with n and it tends to zero as n tends to infinite.

Notice that for $x\in[\sqrt{1/4+2d-\sqrt{d}},\frac{1}{2})$ we have that $p(x,1/2-\sqrt{d})>p(x,1/2+\sqrt{d})>\frac{1}{2}$ which implies that $p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))<p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))$ always holds, since it does not depend on n.

Thus

$$\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]^{\frac{n-1}{2}}<1\quad\text{and}\quad\lim_{n\to\infty}\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]^{\frac{n-1}{2}}=0.$$

Since we have that $\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}>0$ we also have that $[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}]^{\frac{n-1}{2}}\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}+\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}>0$ will hold for large values of n.

Similarly we could show that for $x\in(1/2,1-\sqrt{1/4+2d-\sqrt{d}}]$ we have $\frac{\partial P_n(x,\tilde{\sigma}^D)}{x}<0$ which completes the proof of the lemma. □

Proof of Lemma 5. From the last lemma we know that $\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x}>0$ for $x\in[\sqrt{1/4+2d-\sqrt{d}},\frac{1}{2})$ if and only if

$$\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]^{\frac{n-1}{2}}\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}+\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}>0$$

which can also be written as

$$n>2\frac{\ln\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}+1$$

because we have that $\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}<0$ and we also have that

$$\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}<1\quad\text{implies}\quad\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]<0.$$

For $x=1/2$ to be a local maximum we need to have $\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x}\geqslant 0$ for $x=1/2-\varepsilon$ where $\varepsilon>0$ and $\varepsilon\to 0$.

Thus we have to compute

$$\lim_{x\to 1/2}2\frac{\ln\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}+1.$$

Notice that when x approaches $\frac{1}{2}$ we have that

$$-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\to 1\quad\text{and}\quad\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\to 1,$$

thus we may apply l'Hospital's rule and we obtain that

$$\lim_{x\to 1/2}\frac{\ln\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}=\lim_{x\to 1/2}\frac{\left(-\frac{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}\right)\frac{\partial\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\partial x}}{\frac{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}\frac{\partial\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}{\partial x}}.$$

Observe that since for $x\in(1/2-\sqrt{d},1/2+\sqrt{d})$ we have that

$$p\left(x,\frac{1}{2}-\sqrt{d}\right)=1-\frac{x+\frac{1}{2}-\sqrt{d}}{2}-\frac{d}{2(\frac{1}{2}-\sqrt{d}-x)}\quad\text{and}\quad p\left(x,\frac{1}{2}+\sqrt{d}\right)=\frac{x+\frac{1}{2}+\sqrt{d}}{2}+\frac{d}{2(\frac{1}{2}+\sqrt{d}-x)},$$

then

$$\frac{\partial p(x,\frac{1}{2}-\sqrt{d})}{\partial x}=-\frac{1}{2}-\frac{d}{2(\frac{1}{2}-\sqrt{d}-x)^2}\quad\text{and}\quad\frac{\partial p(x,\frac{1}{2}+\sqrt{d})}{\partial x}=\frac{1}{2}+\frac{d}{2(\frac{1}{2}+\sqrt{d}-x)^2}$$

which implies that

$$\left(-\frac{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}\right)=\frac{1+\frac{d}{(\frac{1}{2}-\sqrt{d}-x)^2}}{1+\frac{d}{(\frac{1}{2}+\sqrt{d}-x)^2}}\to 1$$

and

$$\frac{\partial\left(-\frac{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\partial x}=\frac{\frac{2d}{(\frac{1}{2}+\sqrt{d}-x)^3}-\frac{2d}{(\frac{1}{2}-\sqrt{d}-x)^3}+\frac{2d^2}{(\frac{1}{2}+\sqrt{d}-x)^2(\frac{1}{2}-\sqrt{d}-x)^2}\left(\frac{1}{(\frac{1}{2}+\sqrt{d}-x)}-\frac{1}{(\frac{1}{2}-\sqrt{d}-x)}\right)}{\left(1+\frac{d}{(\frac{1}{2}-\sqrt{d}-x)^2}\right)^2}\to\frac{2}{\sqrt{d}}.$$

Similarly we find that

$$\frac{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}\to\frac{\frac{1}{4}-d}{\frac{1}{4}-d}=1$$

and

$$\begin{aligned}&\frac{\partial\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}{\partial x}\\&\quad=\frac{(1-2p(x,1/2-\sqrt{d}))\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\\&\qquad-\frac{p(x,1/2+\sqrt{d})(1-2p(x,1/2+\sqrt{d}))\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})^2(1-p(x,1/2+\sqrt{d}))^2}\to\frac{4\sqrt{d}}{\frac{1}{4}-d}\end{aligned}$$

because we have that

$$\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}\to -1;\qquad\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}\to +1$$

and

$$1-2p(x,1/2-\sqrt{d})\to -2\sqrt{d};\qquad 1-2p(x,1/2+\sqrt{d})\to -2\sqrt{d}.$$

Therefore, we obtain that

$$\lim_{x\to 1/2}2\frac{\ln\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}+1=2\left(\frac{1-4d}{8d}\right)+1=\frac{1}{4d}.$$

Furthermore, we compute the sign of

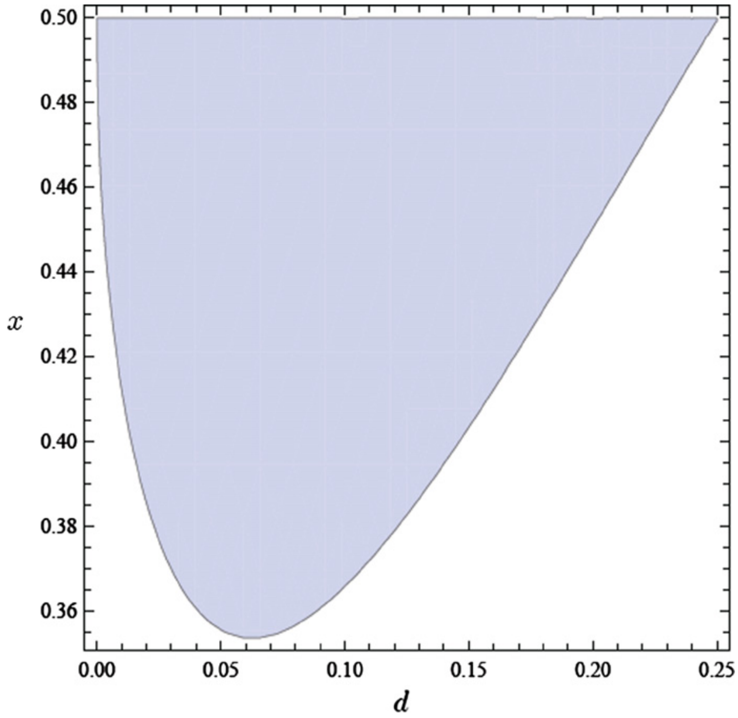

Fig. 6. The shaded area represents $x \in (\sqrt{\frac{1}{4} + 2d - \sqrt{d}}, \frac{1}{2})$.

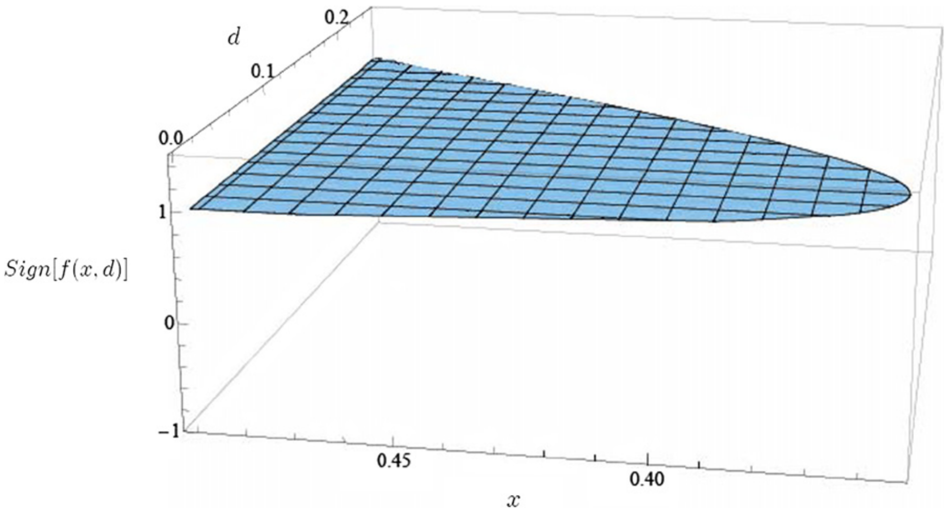

Fig. 7. The sign of the derivative of the payoff function of the advantaged candidate computed by *Mathematica* is always positive for $x \in (\sqrt{\frac{1}{4} + 2d - \sqrt{d}}, \frac{1}{2})$.

$$\frac{\partial \frac{\ln\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]}}{\partial x} \quad \text{for all } x \in \left(\sqrt{\frac{1}{4} + 2d - \sqrt{d}}, \frac{1}{2}\right)$$

and for all $d \in (0, \frac{1}{4})$ using *Mathematica* and we get that it is positive (see Figs. 6 and 7).

This implies that if

$$n > \frac{\ln\left(-\frac{\frac{\partial p(x,1/2+\sqrt{d})}{\partial x}}{\frac{\partial p(x,1/2-\sqrt{d})}{\partial x}}\right)}{\ln\left[\frac{p(x,1/2-\sqrt{d})(1-p(x,1/2-\sqrt{d}))}{p(x,1/2+\sqrt{d})(1-p(x,1/2+\sqrt{d}))}\right]} + 1 \quad \text{for } x \to \frac{1}{2},$$

then the inequality should hold for all $x \in (\sqrt{\frac{1}{4} + 2d - \sqrt{d}}, \frac{1}{2})$ as well.

Thus, $x = \frac{1}{2}$ is a global maximum if and only if $n \geqslant \frac{1}{4d}$. □

Proof of Proposition 2. Since we analyze a two player constant sum game with continuous payoffs, from extension of the minimax theorem (von Neumann, 1928) for continuous payoff functions by Glicksberg (1952), we know that it must be the case that there is a unique equilibrium payoff for each player. This unique equilibrium payoff is given by $P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$ for player A and $1 - P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$ for player D.

We will now prove that $\tilde{\sigma} = \{\tilde{\sigma}^A, \tilde{\sigma}^D\}$ is the unique equilibrium by contradiction. Assume there is another equilibrium $\breve{\sigma} = \{\breve{\sigma}^A, \breve{\sigma}^D\}$ with $\breve{\sigma} \neq \tilde{\sigma}$.

We have previously shown that $x = \frac{1}{2}$ is the unique best response of A when D selects $\tilde{\sigma}^D$. Therefore, for $\breve{\sigma} \neq \tilde{\sigma}$ it must be the case that $\breve{\sigma}^D \neq \tilde{\sigma}^D$. First suppose that $\breve{\sigma}^D$ assigns positive probability only to $y = \frac{1}{2} - \sqrt{d}$ and $y = \frac{1}{2} + \sqrt{d}$. We will show that in equilibrium D can never choose a strategy $\hat{\sigma}^D = (\frac{1}{2} - \sqrt{d}$ w.p. λ; $\frac{1}{2} + \sqrt{d}$ w.p. $1-\lambda)$, where $\lambda \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. When $\lambda = 0$ or $\lambda = 1$, $\hat{\sigma}^D$ is a pure strategy; we have already shown that it is impossible for D to play a pure strategy in equilibrium. Now, when $\lambda \in (0, \frac{1}{2})$ (the proof for $\lambda \in (\frac{1}{2}, 1)$ is symmetric) we observe the following. Since $\tilde{\sigma} = \{\tilde{\sigma}^A, \tilde{\sigma}^D\}$ constitutes an equilibrium $\frac{\partial P_n(x, \tilde{\sigma}^D)}{\partial x} > 0$ should hold for any $x \in [\sqrt{1/4 + 2d - \sqrt{d}}, \frac{1}{2})$. But $\frac{\partial P_n(x, \tilde{\sigma}^D)}{\partial x} > 0$ implies:

$$\left[\frac{p(x, 1/2 - \sqrt{d})(1 - p(x, 1/2 - \sqrt{d}))}{p(x, 1/2 + \sqrt{d})(1 - p(x, 1/2 + \sqrt{d}))}\right]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 - \sqrt{d})}{\partial x} + \frac{\partial p(x, 1/2 + \sqrt{d})}{\partial x} > 0.$$

If we compute $\frac{\partial P_n(x, \hat{\sigma}^D)}{\partial x}$ for $x \in [\sqrt{1/4 + 2d - \sqrt{d}}, \frac{1}{2})$ we find that:

$$\frac{\partial P_n(x, \hat{\sigma}^D)}{\partial x} = n \binom{n-1}{\frac{n-1}{2}} \bigg[\big[\lambda p(x, 1/2 - \sqrt{d})\big(1 - p(x, 1/2 - \sqrt{d})\big)\big]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 - \sqrt{d})}{\partial x} + (1-\lambda)\big[p(x, 1/2 + \sqrt{d})\big(1 - p(x, 1/2 + \sqrt{d})\big)\big]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 + \sqrt{d})}{\partial x} \bigg].$$

Therefore for $\frac{\partial P_n(x, \hat{\sigma}^D)}{\partial x} > 0$ to hold for any $x \in [\sqrt{1/4 + 2d - \sqrt{d}}, \frac{1}{2})$ we must have:

$$\frac{\lambda}{1-\lambda}\left[\frac{p(x, 1/2 - \sqrt{d})(1 - p(x, 1/2 - \sqrt{d}))}{p(x, 1/2 + \sqrt{d})(1 - p(x, 1/2 + \sqrt{d}))}\right]^{\frac{n-1}{2}} \frac{\partial p(x, 1/2 - \sqrt{d})}{\partial x} + \frac{\partial p(x, 1/2 + \sqrt{d})}{\partial x} > 0$$

which obviously holds for any $\lambda \in (0, \frac{1}{2})$. Moreover we observe that for $x = \frac{1}{2}$ we have $\frac{\partial P_n(x, \hat{\sigma}^D)}{\partial x}\big|_{x=\frac{1}{2}} > 0$ since $\frac{\lambda}{1-\lambda}(-1) + 1 > 0$ (this is due to the fact that $p(1/2, 1/2 - \sqrt{d}) = p(1/2, 1/2 + \sqrt{d})$ and $-\frac{\partial p(1/2, 1/2 - \sqrt{d})}{\partial x} = \frac{\partial p(1/2, 1/2 - \sqrt{d})}{\partial x} = 1$).

Finally note that, $P_n(x, \hat{\sigma}^D)$ is increasing in x for any $x < \sqrt{1/4 + 2d - \sqrt{d}}$ and decreasing in x for any $x > 1 - \sqrt{1/4 + 2d - \sqrt{d}}$ following the same reasoning as the one we gave in the $P_n(x, \tilde{\sigma}^D)$ case. The latter suggests that $\arg\max_{x \in [0,1]} P_n(x, \hat{\sigma}^D) \subset (\frac{1}{2}, 1 - \sqrt{1/4 + 2d - \sqrt{d}}]$.

Therefore, one can trivially show that for any possible mixture among the elements of $\arg\max_{x \in [0,1]} P_n(x, \hat{\sigma}^D)$, the best response of D is a mixture of policies that belongs to $(1/2 - \sqrt{d}, 1 - \sqrt{1/4 + 2d - \sqrt{d}} - \sqrt{d}]$. This concludes our argument about D never choosing a strategy $\hat{\sigma}^D = (\frac{1}{2} - \sqrt{d}$ w.p. λ; $\frac{1}{2} + \sqrt{d}$ w.p. $1-\lambda)$, where $\lambda \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, in equilibrium.

We now know that if another equilibrium $\breve{\sigma} = \{\breve{\sigma}^A, \breve{\sigma}^D\}$ exists, then $\breve{\sigma}^D$ must be such that D assigns a positive probability to a set of policies different than just $\frac{1}{2} - \sqrt{d}$ and $\frac{1}{2} + \sqrt{d}$. Now consider that D plays $\breve{\sigma}^D$ and A chooses the pure strategy $x = \frac{1}{2}$ (which is equivalent to $\tilde{\sigma}^A$). We know that $P_n(\tilde{\sigma}^A, y)$ is minimized for $y = \frac{1}{2} - \sqrt{d}$ and for $\frac{1}{2} + \sqrt{d}$ (or when D mixes between these two policies). Since $\breve{\sigma}^D$ is such that D assigns a positive probability to a set of policies different than $\frac{1}{2} - \sqrt{d}$ and $\frac{1}{2} + \sqrt{d}$ we must have $P_n(\tilde{\sigma}^A, \breve{\sigma}^D) > P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$. On the other hand, since $\breve{\sigma} = \{\breve{\sigma}^A, \breve{\sigma}^D\}$ is an equilibrium we must have that $P_n(\breve{\sigma}^A, \breve{\sigma}^D) \geqslant P_n(\tilde{\sigma}^A, \breve{\sigma}^D)$. These two inequalities imply that $P_n(\breve{\sigma}^A, \breve{\sigma}^D) > P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$. But this is impossible because we know that the equilibrium payoff of this game is unique and equal to $P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$. Therefore, no other equilibrium but $\tilde{\sigma} = \{\tilde{\sigma}^A, \tilde{\sigma}^D\}$ exists when $n \geqslant \frac{1}{4d}$. □

Proof of Proposition 3. We will prove first (a) then (c) and finally (b).

(a) If A is using a pure strategy it has to be $x \neq 1/2$ and in that case D has a unique best response ($y = x - \sqrt{d}$ if $x > \frac{1}{2}$ and $y = x + \sqrt{d}$ if $x < \frac{1}{2}$). Since we know that a pure strategy equilibrium does not exist, in equilibrium we cannot have $x \neq \frac{1}{2}$. Otherwise, if candidate A chooses $x = 1/2$ from Lemma 2 we know that D's best response is $\tilde{\sigma}^D$. But we know that $x = \frac{1}{2}$ is not a best response for A against $\tilde{\sigma}^D$ for $n < \frac{1}{4d}$. Thus, it has to be the case that A must be using a mixed strategy as well.

(c) Suppose $P_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) < P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$. Since, $\boldsymbol{\sigma}$ is an equilibrium we should have $P_n(\tilde{\sigma}^A, \boldsymbol{\sigma}^D) \leqslant P_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D)$ and therefore $P_n(\tilde{\sigma}^A, \boldsymbol{\sigma}^D) < P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$. But we know from Lemma 2 that $\tilde{\sigma}^D \in \arg\min P_n(\tilde{\sigma}^A, \sigma^D)$. Therefore, $P_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) \geqslant$

$P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$ and consequently $Q_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) \leqslant Q_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$. We moreover observe that for any $x \in [0, \frac{1}{2}]$ we have $Q_n(x, \frac{1}{2} - \sqrt{d}) \geqslant 0$ and $Q_n(x, \frac{1}{2} + \sqrt{d}) \geqslant Q_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$ (the symmetric holds for any $x \in (\frac{1}{2}, 1]$). Since $Q_n(x, \tilde{\sigma}^D) = \frac{1}{2} Q_n(x, \frac{1}{2} - \sqrt{d}) + \frac{1}{2} Q_n(x, \frac{1}{2} + \sqrt{d})$ we can easily conclude that $Q_n(x, \tilde{\sigma}^D) \geqslant \frac{1}{2} Q_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$ for any $x \in [0, 1]$. Therefore $Q_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) \geqslant \frac{1}{2} Q_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$ and consequently $P_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D) \leqslant \frac{1}{2} + \frac{1}{2} P_n(\tilde{\sigma}^A, \tilde{\sigma}^D)$.

(b) From the above we observe that $P_n(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^D)$ is always strictly smaller than 1. Then it must be the case that $P_n(s^A, \boldsymbol{\sigma}^D) < 1$. Now suppose that $s^A \leqslant s^D$. We know that $\frac{\partial P_n(x,y)}{\partial x} \geqslant 0$ for any $x \leqslant y$ and that $\frac{\partial P_n(x,y)}{\partial x} > 0$ whenever $x < y$ and $P_n(x, y) < 1$. Thus, $\frac{\partial P_n(x,y)}{\partial x}\big|_{x=s^A} \geqslant 0$ for any $y \in [s^D, S^D]$ and $\frac{\partial P_n(x,y)}{\partial x}\big|_{x=s^A} > 0$ for $y \in [S^D - \varepsilon, S^D]$ for some positive ε (otherwise $P_n(s^A, \boldsymbol{\sigma}^D)$ should be equal to 1). This implies that $\frac{\partial P_n(x,\boldsymbol{\sigma}^D)}{\partial x}\big|_{x=s^A} > 0$. In other words s^A is not a best response for A to $\boldsymbol{\sigma}^D$ and cannot be part of $\boldsymbol{\sigma}^A$. Therefore we should have $s^A > s^D$ and similarly $S^A < S^D$. □

Proof of Proposition 4. (a) Notice that if $d \geqslant \phi(\frac{1}{2})$ then $d > \phi(|x_i - 1/2|) > 0$ for all $x_i \in [0, 1]$ because $\phi(.)$ is strictly increasing. Therefore we must have that $U_i(1/2) = d - \phi(|x_i - 1/2|) > 0 \geqslant U_i(x) = -\phi(|x_i - x|)$ for all $x_i \in [0, 1]$ and all $x \in [0, 1]$. This implies that A can always choose the pure strategy $\frac{1}{2}$ and win with certainty independently of the strategy of D. (Pure strategy equilibrium.)

(b) Following the previous argument if $d < \phi(\frac{1}{2})$ for any pure strategy of A, there is a best response for D (different from A's) which guarantees D a positive payoff. But the best response for A would be to copy D's strategy and win with certainty. Thus, the equilibrium must be in mixed strategies.

(c) By generalizing the utility functions of our voters, the only thing that we change in the model is actually the cutpoint $\hat{x}(x, y)$. Given the generality of $\phi(.)$ we cannot have a certain functional form of $\hat{x}(x, y)$. Despite this complication we know that $\hat{x}(x, y)$ is twice differentiable everywhere except at $x = y$, that for any $x \neq y$ it is unique and that $|x - \hat{x}(x, y)| > |y - \hat{x}(x, y)|$ (Groseclose, 2001). Therefore, our formulation of $p(x, y)$ and $P_n(x, y)$ implies that still both $p(x, y)$ and $P_n(x, y)$ are continuous in $[0, 1]^2$ and differentiable in the same domain apart from the combinations of x's and y's such that $\hat{x}(x, y)$ becomes zero or one (exactly like the quadratic case). Remember that $\hat{x}(x, y)$ is such that $d - \phi(|\hat{x}(x, y) - x|) = -\phi(|\hat{x}(x, y) - y|)$. Therefore, differentiating both sides of the latter equality with respect to x gives us:

$$\frac{\partial \hat{x}(x, y)}{\partial x} = \frac{sgn(\hat{x}(x, y) - x)\phi'(|\hat{x}(x, y) - x|)}{sgn(\hat{x}(x, y) - x)\phi'(|\hat{x}(x, y) - x|) - sgn(\hat{x}(x, y) - y)\phi'(|\hat{x}(x, y) - y|)},$$

and differentiating the above once more with respect to x results in:

$$\frac{\partial^2 \hat{x}(x, y)}{\partial x^2} = \frac{[\phi'(|\hat{x}(x, y) - x|)]^2 \phi''(|\hat{x}(x, y) - y|) - [\phi'(|\hat{x}(x, y) - y|)]^2 \phi''(|\hat{x}(x, y) - x|)}{[sgn(\hat{x}(x, y) - x)\phi'(|\hat{x}(x, y) - x|) - sgn(\hat{x}(x, y) - y)\phi'(|\hat{x}(x, y) - y|)]^3}.$$

Observe that for $y = \frac{1}{2} - \phi^{-1}(d)$ and $x = \frac{1}{2}$ we get $\hat{x}(\frac{1}{2}, \frac{1}{2} - \phi^{-1}(d)) = \frac{1}{2} - \phi^{-1}(d)$ and, equivalently, for $y = \frac{1}{2} + \phi^{-1}(d)$ and $x = \frac{1}{2}$ we get $\hat{x}(\frac{1}{2}, \frac{1}{2} + \phi^{-1}(d)) = \frac{1}{2} + \phi^{-1}(d)$. Therefore, we have $\frac{\partial \hat{x}(x, \frac{1}{2} - \phi^{-1}(d))}{\partial x}\big|_{x=\frac{1}{2}} = \frac{\partial \hat{x}(x, \frac{1}{2} + \phi^{-1}(d))}{\partial x}\big|_{x=\frac{1}{2}} = 1$ and $\frac{\partial^2 \hat{x}(x, \frac{1}{2} - \phi^{-1}(d))}{\partial x^2}\big|_{x=\frac{1}{2}} = -\frac{\partial^2 \hat{x}(x, \frac{1}{2} + \phi^{-1}(d))}{\partial x^2}\big|_{x=\frac{1}{2}} = -\frac{\phi''(0)}{\phi'(\phi^{-1}(d))}$. This observation will be crucial for our proof.

Let's show first that $\tilde{\sigma}^D = (\frac{1}{2} - \phi^{-1}(d)$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \phi^{-1}(d)$ w.p. $\frac{1}{2})$ is a best response to $x = \frac{1}{2}$. We observe that when A plays $x = \frac{1}{2}$ then if the ideal policy of a voter is such that $x_i \in (\frac{1}{2} - \phi^{-1}(d), \frac{1}{2} + \phi^{-1}(d))$, A gets this voter's vote independently of the strategy of D. This is because this voter cannot get positive utility from D and because $d - \phi(|x_i - \frac{1}{2}|) > 0$ implies that a voter with ideal policy $x_i \in (\frac{1}{2} - \phi^{-1}(d), \frac{1}{2} + \phi^{-1}(d))$ gets positive utility from A. In other words, when A plays $\frac{1}{2}$ the cutpoint cannot be in $(\frac{1}{2} - \phi^{-1}(d), \frac{1}{2} + \phi^{-1}(d))$. We have moreover seen that $\hat{x}(\frac{1}{2}, \frac{1}{2} - \phi^{-1}(d)) = \frac{1}{2} - \phi^{-1}(d)$ and that $\hat{x}(\frac{1}{2}, \frac{1}{2} + \phi^{-1}(d)) = \frac{1}{2} + \phi^{-1}(d)$. Therefore, the winning probability of D is maximized when $y = \frac{1}{2} - \phi^{-1}(d)$ or when $y = \frac{1}{2} + \phi^{-1}(d)$ (or when D mixes between these two policies) for any n; $\tilde{\sigma}^D = (\frac{1}{2} - \phi^{-1}(d)$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \phi^{-1}(d)$ w.p. $\frac{1}{2})$ is a best response of D to A playing the pure strategy $x = \frac{1}{2}$ for any n.

Now if D plays $\tilde{\sigma}^D = (\frac{1}{2} - \phi^{-1}(d)$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \phi^{-1}(d)$ w.p. $\frac{1}{2})$ we observe the following. The payoff function of A, $P_n(x, \tilde{\sigma}^D)$ is continuous and twice differentiable everywhere apart from four points. As in the quadratic case, $P_n(x, \tilde{\sigma}^D)$ is a linear combination of $P_n(x, \frac{1}{2} - \phi^{-1}(d))$ and $P_n(x, \frac{1}{2} + \phi^{-1}(d))$ and these two latter expressions are respectively differentiable everywhere apart from the points where $\hat{x}(x, \frac{1}{2} - \phi^{-1}(d))$ and $\hat{x}(x, \frac{1}{2} + \phi^{-1}(d))$ become 1 or 0. Therefore, to find these four points where $P_n(x, \tilde{\sigma}^D)$ is not differentiable we have to compute $d - \phi(|1 - x|) = -\phi(|1 - \frac{1}{2} + \phi^{-1}(d)|)$, $d - \phi(|0 - x|) = -\phi(|0 - \frac{1}{2} + \phi^{-1}(d)|)$, $d - \phi(|1 - x|) = -\phi(|1 - \frac{1}{2} - \phi^{-1}(d)|)$ and $d - \phi(|0 - x|) = -\phi(|0 - \frac{1}{2} - \phi^{-1}(d)|)$. We find that these four points ranked in an increasing manner are: $1 - \phi^{-1}(d + \phi(\frac{1}{2} + \phi^{-1}(d)))$, $\phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d)))$, $1 - \phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d)))$ and $\phi^{-1}(d + \phi(\frac{1}{2} + \phi^{-1}(d)))$. Using the same reasoning as in the quadratic case it is apparent that $P_n(x, \tilde{\sigma}^D)$ is increasing in $[0, \phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d))))$ and decreasing in $(1 - \phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d))), 1]$ for any n.

So we only have to focus on $[\phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d))), 1 - \phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d)))]$ in order to investigate the optimal behavior of A when D plays $\tilde{\sigma}^D = (\frac{1}{2} - \phi^{-1}(d)$ w.p. $\frac{1}{2}$; $\frac{1}{2} + \phi^{-1}(d)$ w.p. $\frac{1}{2})$. Since $P_n(x, \tilde{\sigma}^D)$ is twice differentiable in $(\phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d))), 1 - \phi^{-1}(d + \phi(\frac{1}{2} - \phi^{-1}(d))))$ we get:

$$\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x} = \frac{n}{2}\binom{n-1}{\frac{n-1}{2}}\Bigg[\big[p\big(x,1/2-\phi^{-1}(d)\big)\big(1-p\big(x,1/2-\phi^{-1}(d)\big)\big)\big]^{\frac{n-1}{2}}\frac{\partial p(x,1/2-\phi^{-1}(d))}{\partial x} + \big[p\big(x,1/2+\phi^{-1}(d)\big)\big(1-p\big(x,1/2+\phi^{-1}(d)\big)\big)\big]^{\frac{n-1}{2}}\frac{\partial p(x,1/2+\phi^{-1}(d))}{\partial x}\Bigg]$$

which is positive for $x \in (\phi^{-1}(d+\phi(\frac{1}{2}-\phi^{-1}(d))), \frac{1}{2})$ (negative for $x \in (\frac{1}{2}, 1-\phi^{-1}(d+\phi(\frac{1}{2}-\phi^{-1}(d)))))$ if and only if:

$$\left[\frac{p(x,1/2-\phi^{-1}(d))(1-p(x,1/2-\phi^{-1}(d)))}{p(x,1/2+\phi^{-1}(d))(1-p(x,1/2+\phi^{-1}(d)))}\right]^{\frac{n-1}{2}}\frac{\partial p(x,1/2-\phi^{-1}(d))}{\partial x} + \frac{\partial p(x,1/2+\phi^{-1}(d))}{\partial x} > 0.$$

Since $\frac{\partial p(x,1/2-\phi^{-1}(d))}{\partial x} = -\frac{\partial \hat{x}(x,\frac{1}{2}-\phi^{-1}(d))}{\partial x} < 0$ and $\frac{\partial p(x,1/2+\phi^{-1}(d))}{\partial x} = \frac{\partial \hat{x}(x,\frac{1}{2}+\phi^{-1}(d))}{\partial x} > 0$ with the same reasoning as in the quadratic case we conclude that for sufficiently high (but finite) values of n, $\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x} > 0$ for $x \in (\phi^{-1}(d+\phi(\frac{1}{2}-\phi^{-1}(d))), \frac{1}{2})$.

In the quadratic case we studied in detail the behavior of $P_n(x,\tilde{\sigma}^D)$ when $x=\frac{1}{2}$. Here, due to the lack of the specific functional form of $\phi()$, we have to adopt a more general approach to see what is going on in the center of the policy space. We will show that for sufficiently high (but finite) values of n, $x=\frac{1}{2}$ becomes a local maximum of $P_n(x,\tilde{\sigma}^D)$. This result combined with the fact that sufficiently high (but finite) values of n make $\frac{\partial P_n(x,\tilde{\sigma}^D)}{\partial x} > 0$ for $x \in (\phi^{-1}(d+\phi(\frac{1}{2}-\phi^{-1}(d))), \frac{1}{2})$ are sufficient to prove part (c) of the proposition.

For $x=\frac{1}{2}$ to be a local maximum of $P_n(x,\tilde{\sigma}^D)$ it must be the case that $\frac{\partial^2 P_n(x,\tilde{\sigma}^D)}{\partial x^2}\big|_{x=\frac{1}{2}} \leqslant 0$. We observe that:

$$\begin{aligned}\frac{\partial^2 P_n(x,\tilde{\sigma}^D)}{\partial x^2} = \frac{n}{2}\binom{n-1}{\frac{n-1}{2}}\Bigg[&\big[p\big(x,1/2-\phi^{-1}(d)\big)\big(1-p\big(x,1/2-\phi^{-1}(d)\big)\big)\big]^{\frac{n-1}{2}}\frac{\partial^2 p(x,1/2-\phi^{-1}(d))}{\partial x^2}\\ &+\frac{n-1}{2}\big[p\big(x,1/2-\phi^{-1}(d)\big)\big(1-p\big(x,1/2-\phi^{-1}(d)\big)\big)\big]^{\frac{n-1}{2}-1}\\ &\times\big[1-2p\big(x,1/2-\phi^{-1}(d)\big)\big]\left(\frac{\partial p(x,1/2-\phi^{-1}(d))}{\partial x}\right)^2\\ &+\big[p\big(x,1/2+\phi^{-1}(d)\big)\big(1-p\big(x,1/2+\phi^{-1}(d)\big)\big)\big]^{\frac{n-1}{2}}\frac{\partial^2 p(x,1/2+\phi^{-1}(d))}{\partial x^2}\\ &+\frac{n-1}{2}\big[p\big(x,1/2+\phi^{-1}(d)\big)\big(1-p\big(x,1/2+\phi^{-1}(d)\big)\big)\big]^{\frac{n-1}{2}-1}\\ &\times\big[1-2p\big(x,1/2+\phi^{-1}(d)\big)\big]\left(\frac{\partial p(x,1/2+\phi^{-1}(d))}{\partial x}\right)^2\Bigg].\end{aligned}$$

Using the fact that $\frac{\partial^2 p(x,1/2-\phi^{-1}(d))}{\partial x^2} = -\frac{\partial^2 \hat{x}(x,\frac{1}{2}-\phi^{-1}(d))}{\partial x^2}$ and that $\frac{\partial^2 p(x,1/2+\phi^{-1}(d))}{\partial x^2} = \frac{\partial^2 \hat{x}(x,\frac{1}{2}+\phi^{-1}(d))}{\partial x^2}$ we conclude that $\frac{\partial^2 P_n(x,\tilde{\sigma}^D)}{\partial x^2}\big|_{x=\frac{1}{2}} \leqslant 0$ if and only if $n \geqslant \frac{\phi''(0)[\frac{1}{4}-(\phi^{-1}(d))^2]}{\phi'(\phi^{-1}(d))\phi^{-1}(d)} + 1$. Observe that due to the assumptions regarding $\phi()$, $\frac{\phi''(0)[\frac{1}{4}-(\phi^{-1}(d))^2]}{\phi'(\phi^{-1}(d))\phi^{-1}(d)}$ is always positive and finite valued. So for sufficiently high (but finite) values of n we actually have that $x=\frac{1}{2}$ is a local maximum of $P_n(x,\tilde{\sigma}^D)$. This completes the proof. □

References

Adams, James, 1999. Policy divergence in multicandidate probabilistic spatial voting. Public Choice 100, 103–122.

Ansolabehere, Stephen, Snyder Jr., James M., 2000. Valence politics and equilibrium in spatial election models. Public Choice 103 (June), 327–336.

Aragones, Enriqueta, Palfrey, Thomas R., 2002. Mixed equilibrium in a Downsian model with a favored candidate. J. Econ. Theory 103, 131–161.

Aragones, Enriqueta, Palfrey, Thomas R., 2004. The effect of candidate quality on electoral equilibrium: An experimental study. Amer. Polit. Sci. Rev. 98 (1), 77–90.

Aragones, Enriqueta, Palfrey, Thomas R., 2005. Spatial competition between two candidates of different quality: The effects of candidate ideology and private information. In: Austen-Smith, D., Duggan, J. (Eds.), Social Choice and Strategic Decisions. Springer, Berlin.

Berger, Mark M., Munger, Michael C., Potthoff, Richard F., 2000. With uncertainty, the Downsian model predicts divergence. J. Theoretical Politics 12 (April), 262–268.

Bernhardt, Daniel, Ingberman, Daniel, 1985. Candidate reputations and the incumbency effect. J. Public Econ. 27, 47–67.

Glicksberg, I.L., 1952. A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points. Proc. Amer. Math. Soc. 38, 170–174.

Groseclose, Timothy, 2001. A model of candidate location when one candidate has a valence advantage. Amer. J. Polit. Sci. 45 (October), 862–886.

Hummel, Patrick, 2010. On the nature of equilibria in a Downsian model with candidate valence. Games Econ. Behav. 70 (2), 425–445.

Kiewiet, D. Roderick, 1983. Macroeconomics and Micropolitics. University of Chicago Press, Chicago.

Kiewiet, D. Roderick, Zeng, Langche, 1993. An analysis of congressional career decisions, 1947–1986. Amer. Polit. Sci. Rev. 87, 928–941.

Kirstein, Roland, von Wagenheim, Georg, 2010. A generalized Condorcet Jury Theorem with two independent probabilities of error. MAGKS Discussion Paper Series in Economics.

Londregan, John, Romer, Thomas, 1993. Polarization, incumbency, and the personal vote. In: Barnett, W.A., Hinich, M.J., Schofield, N.J. (Eds.), Political Economy: Institutions, Competition, and Representation. Cambridge University Press, Cambridge, pp. 355–377.

Stokes, Donald E., 1963. Spatial models of party competition. Amer. Polit. Sci. Rev. 57, 368–377.

von Neumann, J., 1928. Zur Theorie der Gesellschaftsspiele. Math. Ann. 100 (1), 295–320. English translation: On the theory of games of strategy, in: Tucker, A.W., Luce, R.D. (Eds.), Contributions to the Theory of Games, vol. 4, 1959, pp. 13–42.